NOTES ON NOV 22, 2012

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I had a tight agenda for this class because there were a couple of questions on the homework that people asked me about.

First, the questions about how to prove the generating functions $B(x,u) = \sum_{n\geq 0}\sum_{k=1}^{n} S(n,k)u^k \frac{x^n}{n!}$ is equal to $e^{u(e^x-1)}$. I had talked with a few people after class and they wanted to know how to do this problem because it wasn't exactly like what I had done in class for $B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$ to show that $B(x) = e^{e^x-1}$. I said, well if you worked on this problem you should have all found that

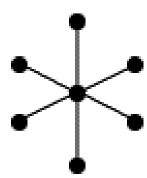
$$\frac{\partial}{\partial x}B(x,u) = uB(x,u) + u\frac{\partial}{\partial u}B(x,u)\ .$$

Then, once you do this, you have more or less shown that $B(x,u) = e^{u(e^x-1)}$ after you also show that

$$\frac{\partial}{\partial x}(e^{u(e^x-1)}) = u(e^{u(e^x-1)}) + u\frac{\partial}{\partial u}(e^{u(e^x-1)})\ .$$

Why? Think about it and I will explain next time why this (plus one or two minor details) proves that $B(x,u) = e^{u(e^x-1)}$.

The next thing I discussed was the number of ways of coloring the spoked graph:



A motion of this graph which preserves the structure can permute any of the six outer vertices but the center vertex must be fixed. Once we know how many cycles of each type there are, we color each of the cycles with k colors. The number of colorings which are fixed by a permutation with d cycles is equal to k^d. But recall that at the beginning of the class we computed a formula for the number of permutations of n with d cycles and this was the unsigned Stirling numbers of the first kind.

In particular we had a table (see the notes from Sept 13-18 for a table of the signed Stirling numbers of the first kind but only up to $n = 4$). We also had a formula from the first homework assignment. We also have the last problem from homework #4 and the formula $\sum_{n\geq 0}\sum_{k=1}^{n} s'(n,k)u^k \frac{x^n}{n!} = e^{-u\ log(1-x)}$. I used the computer then to compute the unsigned Stirling numbers assuming that this formula is correct.

```
sage: (u,x) = var('u,x')
sage: taylor(exp(-u*log(1-x)),x,0,10)
1/3628800*(u^10 + 45*u^9 + 870*u^8 + 9450*u^7 + 63273*u^6 + 269325*u^5 +
 723680*u^4 + 1172700*u^3 + 1026576*u^2 + 362880*u)*x^10 + 1/362880*(u^9
 + 36*u^8 + 546*u^7 + 4536*u^6 + 22449*u^5 + 67284*u^4 + 118124*u^3 +
 109584*u^2 + 40320*u)*x^9 + 1/40320*(u^8 + 28*u^7 + 322*u^6 + 1960*u^5
 + 6769*u^4 + 13132*u^3 + 13068*u^2 + 5040*u)*x^8 + 1/5040*(u^7 + 21*u^6
 + 175*u^5 + 735*u^4 + 1624*u^3 + 1764*u^2 + 720*u)*x^7 + 1/720*(u^6 +
 15*u^5 + 85*u^4 + 225*u^3 + 274*u^2 + 120*u)*x^6 + 1/120*(u^5 + 10*u^4
 + 35*u^3 + 50*u^2 + 24*u)*x^5 + 1/24*(u^4 + 6*u^3 + 11*u^2 + 6*u)*x^4
 + 1/6*(u^3 + 3*u^2 + 2*u)*x^3 + 1/2*(u^2 + u)*x^2 + u*x + 1
```

This calculation says that (for instance) there are 15 permutations of 6 with 5 cycles. Because these permutations have one cycle of length 2 and 4 cycles of length 1 we know that there are $\binom{6}{2}$ possible permutations because this is the number of ways of choosing two elements to make a 2 cycle.

Once we know that a permutation has d cycles for the 6 outer vertices and one cycle of length 1 for the center vertex, then there are k^{d+1} ways of coloring the graph so that it is fixed by that permutation.

If we were to write down the formula for the colorings of the graph above it would be

$$\frac{1}{6!}\sum_{d=1}^{6} s'(6,d)k^{d+1} = \frac{1}{720}(k^7 + 15k^6 + 85k^5 + 225k^4 + 274k^3 + 120k^2)$$

This argument is quite general and it also says that in fact that the number of colorings of the spoke graph where there are n spokes coming off of a center vertex is given by

$$\frac{1}{n!}\sum_{d=1}^{n} s'(n,d)k^{d+1} \ .$$

I should remind you that in the first homework assignment we showed that $(k)^{(n)} = k(k+1)(k+2)\cdots(k+(n-1)) = \sum_{d=1}^{n} s'(n,d)k^d$, hence the number of colorings of the spoke graph with n spokes is equal to

$$\frac{1}{n!}\sum_{d=1}^{n} s'(n,d)k^{d+1} = \frac{k(k)^{(n)}}{n!} \ .$$

We can't really see this factorization on the sage example above because we have to tell the computer to factor each of the coefficients of the series if that is what we want.

```
sage: f = taylor(exp(-u*log(1-x)),x,0,10)
sage: sum(factor(f.coefficient(x,i))*x^i for i in range(11))
1/3628800*(u + 1)*(u + 2)*(u + 3)*(u + 4)*(u + 5)*(u + 6)*(u + 7)*(u
 + 8)*(u + 9)*u*x^10 + 1/362880*(u + 1)*(u + 2)*(u + 3)*(u + 4)*(u +
 5)*(u + 6)*(u + 7)*(u + 8)*u*x^9 + 1/40320*(u + 1)*(u + 2)*(u +
 3)*(u + 4)*(u + 5)*(u + 6)*(u + 7)*u*x^8 + 1/5040*(u + 1)*(u + 2)*(u
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 + 3)*(u + 4)*(u + 5)*(u + 6)*u*x^7 + 1/720*(u + 1)*(u + 2)*(u + 3)*(u
 + 4)*(u + 5)*u*x^6 + 1/120*(u + 1)*(u + 2)*(u + 3)*(u + 4)*u*x^5 +
1/24*(u + 1)*(u + 2)*(u + 3)*u*x^4 + 1/6*(u + 1)*(u + 2)*u*x^3 +
1/2*(u + 1)*u*x^2 + u*x + 1
```

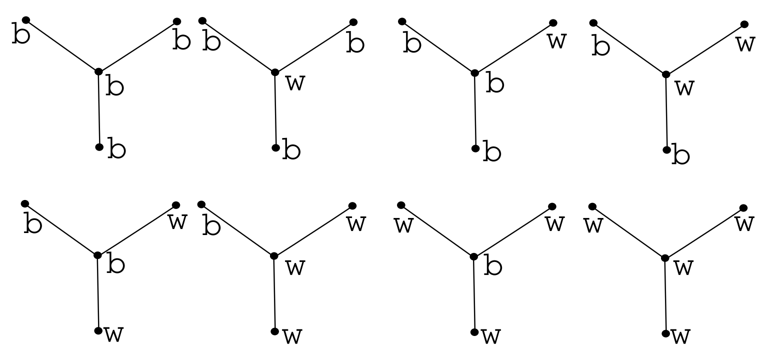
Moreover we can also not only give the formula for a single entry in this sequence, we can use this formula to give the generating function for all possible colorings of all the spoke graphs at the same time.

$$\begin{aligned}
&\text{g.f. for colorings of spoke graph with } n \text{ spokes}\\
&= \sum_{n\geq 0} (\text{number of colorings of } n \text{ spoke graph}) x^n\\
&= \sum_{n\geq 0} \frac{1}{n!} \sum_{d=1}^{n} s'(n,k) k^{d+1} x^n\\
&= k \sum_{n\geq 0} \sum_{d=1}^{n} s'(n,k) k^{d+1} \frac{x^n}{n!}
\end{aligned}$$

You should notice however that this is precisely the generating function $ke^{-klog(1-x)}$ given in the last problem of the homework. So for instance lets say that we set $k = 2$ and look at the computer expansion of this series.

```
sage: taylor(2*exp(-2*log(1-x)),x,0,10)
22*x^10 + 20*x^9 + 18*x^8 + 16*x^7 + 14*x^6 + 12*x^5 + 10*x^4 + 8*x^3
 + 6*x^2 + 4*x + 2
```

This shows that for instance the graph with three spokes coming off of a center vertex can be colored with $k = 2$ colors in 8 different ways so that the colorings are distinctly different. For example:



are all possible colorings of the graph with three spokes coming off of a center vertex. What is very cool is that this homework problem relates the problems in HW #1 part 1 question 2,3 and HW #4 part 1 question 1 and HW # 4 part 2 question 3.

So the next thing that I wanted to discuss was the number of permutations with a given cycle structure. I rushed through the explanation at the end of class last time and I wanted to give a few more details about the formula that I stated very quickly.

Say that we want to know how many permutations there with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc. This means that

$$n = a_1 + 2a_2 + 3a_3 + \cdots$$

There are only a finite number of solutions to this equation for any fixed n. In particular there is one for every partition of n. This is because the lengths of the cycles of the permutation determine a partition (e.g. $(123)(456)(78)(9)$ and $(1)(234)(56)(789)$ both determine the partition $(3, 3, 2, 1)$ by the lengths of their cycles since the order of the cycles is not important).

So lets say that we wanted to count the number of permutations with a_i cycles of length i for $i \geq 1$. In this case we want to find the orbit of the permutation:

$$\pi = (1)(2)\cdots(b_1)(b_1+1, b_1+2)\cdots(b_2-1, b_2)(b_2+1, b_2+2, b_2+3)\cdots(b_3-2, b_3-1, b_3)\cdots$$

where $b_1 = a_1$, $b_2 = a_1 + 2a_2$, $b_3 = a_1 + 2a_2 + 3a_3$, and in general $b_r = \sum_{i=1}^{r} a_i$. This is a permutation with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc.

Now I want to make a procedure which determines another representation of this permutation which is equivalent. That is I want to determine an element g such that $g \circ \pi \circ g^{-1} = \pi$.

- The a_1 cycles of length 1 may be sent to a permutation of the a_1 cycles.
- The a_2 cycles of length 2 are of the form $(i, i+1)$ and they may be sent to a permutation of the cycles and each one may be sent to either a cycle of the form $(j, j+1)$ or $(j+1, j)$.
- The a_3 cycles of length 3 are of the form $(i, i+1, i+2)$ and they may be sent to a permutation of the cycles and each one may be sent to one of the form $(j, j+1, j+2)$, $(j+1, j+2, j)$ or $(j+2, j, j+1)$
- In general, the a_r cycles of length r are of the form $(i, i+1, \ldots, i+r-1)$ they may be sent to a permutation of the a_r cycles and each one may be sent to one of the r different cyclic shifts of the cycles $(i+d, i+d+1, \ldots, i+r-1, i, i+1, \ldots, i+d-1)$.

This describes a procedure for determining one possible permutation g which is in the stabilizer of π. The number of outcomes of the r^{th} step of this procedure is $a_r! r^{a_r}$ since there are $a_r!$ ways of permuting the cycles and r choices for each one of the cycles as to how it is shifted. Hence, by the multiplication principle the number of elements in the stabilizer of π is equal to

$$a_1! 1^{a_1} a_2! 2^{a_2} a_3! 3^{a_3} \cdots = \prod_{r \geq 0} a_r! r^{a_r} \ . \tag{1}$$

Now since the number of elements in the stabilizer of π is equal to equation (1), the number of elements in orbit of this group action is equal to $n!$ divided by the number of elements in the stabilizer,

$$\frac{n!}{\prod_{r\geq 0} a_r! r^{a_r}} \ .$$

The number of elements in the orbit of this element is equal to the number of permutations with a_r cycles of length r for $r \geq 1$.

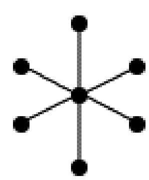
Lets do this for permutations of 6 and verify that it agrees with what we computed for the unsigned Stirling numbers that we computed from the generating function earlier.

permutation π	$\#\{g \in S_6 : g \circ \pi \circ g^{-1} = \pi\}$	$\#\{g \circ \pi \circ g^{-1} : g \in S_6\}$
(1)(2)(3)(4)(5)(6)	$6! = 720$	$720/720 = 1$
(1)(2)(3)(4)(56)	$4!2 = 48$	$720/48 = 15$
(1)(2)(34)(56)	$2!2!2^2 = 16$	$720/16 = 45$
(1)(2)(3)(456)	$3!3 = 18$	$720/18 = 40$
(12)(34)(56)	$3!2^3 = 48$	$720/48 = 15$
(1)(23)(456)	$2 \cdot 3 = 6$	$720/6 = 120$
(1)(2)(3456)	$2!4 = 8$	$720/8 = 90$
(123)(456)	$2!3^2 = 18$	$720/18 = 40$
(12)(3456)	$2 \cdot 4 = 8$	$720/8 = 90$
(1)(23456)	5	$720/5 = 144$
(123456)	6	$720/6 = 120$

We can use this table to compute that $s'(6,5) = 15$, $s'(6,4) = 45 + 40 = 85$, $s'(6,3) = 15 + 120 + 90 = 225$, $s'(6,2) = 40 + 90 + 144 = 274$, $s'(6,1) = 120$.

You might ask (and someone did) if we can use the Stirling numbers to compute the number of permutations with a given number of cycles, then why do we need to know how to find the number of permutations with a given cycle type? If you need to apply Polya's theorem rather than Burnside's Lemma (a formula which contains more detailed information), then you need to know precisely the number of cycles of each type for each of the permutations and not just how many cycles there are. In particular I can ask more pointed questions like the following:

How many colorings of the graph



are there using k colors such that each color is used at most twice?