# NOTES ON NOV 27, 2012

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I answered questions about the homework problems. One of them was the second problem about generating functions. Someone simply asked 'how do you do it?' It is hard to give a hint on this one with out shoving you in the right direction. The problem was to prove a formula (which was part of the problem but I don't remember what it is right now) for the $S(n,k)$. But in problem number (1) you were asked to show that the generating function for $S(n,k)$ is equal to $e^{u(e^x-1)}$. Armed with this piece of information you know that the coefficient of $u^k\frac{x^n}{n!}$ in $e^{u(e^x-1)}$ is $S(n,k)$, but you also have that

$$e^{u(e^x-1)} = \sum_{d\geq 0} u^d \frac{(e^x-1)^d}{d!}$$

Now you should notice that you can use the binomial theorem to expand $(e^x-1)^d = \sum_{i=0}^{d}\binom{d}{i}(-1)^i e^{(d-i)x}$ and now take the coefficient of $u^k\frac{x^n}{n!}$ in the expression you get there. At this point there isn't too much left to do but remember that the coefficient of $\frac{x^n}{n!}$ in $e^{cx}$ is equal to $c^n$.

Then I knew that I wanted to talk a little bit about the first and the third problems in that section. I said in the last class that 'all' you had to do was show that $B(x,u) := 1 + \sum_{n\geq 1}\sum_{k=1}^{n} S(n,k)u^k\frac{x^n}{n!}$ satisfied the differential equation

$$\frac{\partial}{\partial x}B(x,u) = uB(x,u) + u\frac{\partial}{\partial u}B(x,u)$$

and you were done, but that is a little inaccurate. It is the major step of the proof, but there is an argument to be made to verify that you really are done.

The coefficients $S(n,k)$ are defined by the recurrence $S(n+1,k) = S(n,k-1) + kS(n,k)$ for $n \geq 0$ and $k \geq 1$ and the initial conditions that $S(0,0) = 1$ and $S(n,0) = S(0,n) = 0$ for $n > 0$. What you need to do is show that the coefficients in the series for $e^{u(e^x-1)}$ also satisfies the same defining relations.

There are three steps that you need to complete in order to show this. First, let $V(x,u)$ be a function with a taylor series $V(x,u) = \sum_{n,k\geq 0} a_{n,k}u^k\frac{x^n}{n!}$ and show that $V(x,u)$ satisfies

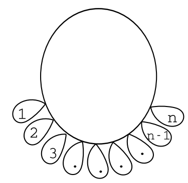
$$\frac{\partial}{\partial x}V(x,u) = uV(x,u) + u\frac{\partial}{\partial u}V(x,u)$$

if and only if

$$a_{n+1,k} = a_{n,k-1} + ka_{n,k}$$

for $n \geq 0$ and $k \geq 1$. This is more or less exactly what you needed to do in order to show that $B(x, u)$ satisfies this equation, but you also need to go backwards. Second you need to show that $e^{u(e^x-1)}$ satisfies this differential equation. This is a relatively easy calculus calculation. Finally, you need to show that the coefficients satisfies the same base case. This amounts to showing $B(x,u)\Big|_{x^0} = e^{u(e^x-1)}\Big|_{x^0}$ and $B(x,u)\Big|_{u^0} = e^{u(e^x-1)}\Big|_{u^0}$. Since, both of these coefficients is equal to 1, you have shown that the coefficients satisfy the same base case and hence $a_{n,k} = S_{n,k}$ for all $n, k \geq 0$.

Next I talked about the formula for necklaces. I drew the picture of a necklace with beads hanging from a chain and I indicated that the motions of the necklace were $R_r$ for $1 \leq r \leq n$ where this means take $r$ beads from the right hand side and move them to the left hand side (note: for convenience I switched directions from the notation I used on November 20, but really this affects nothing significantly).




Notice what happens to bead number $i$ under the action of $R_r$. Bead $i$ is sent to $i + r$; then bead $i + r$ is sent to $i + 2r$; bead $i + 2r$ ends up where $i + 3r$ was located; etc. This will make a cycle of length $d$ when $i + dr$ ends up where bead $i$ currently is. In order for this to happen $dr$ must be a multiple of $n$ (the total number of beads and this cycle will be exactly of length $d$ if $dr = lcm(n, r)$.

There is a well known formula for $lcm(n, r)$ in terms of the greatest common divisor.

**Lemma 1.** *For positive integers $a$ and $b$, $lcm(a, b) = \frac{ab}{gcd(a,b)}$ .*

Take for example the $lcm(10, 12) = 60$, this formula says it should be $10 \cdot 12 = 120$ divided by the $gcd(10, 12) = 2$. I provided a quick proof of this fact just to convince you that it was true by looking at the prime factorizations of $a, b, gcd(a, b)$ and $lcm(a, b)$, but I won't bother to write it down here because it is based on the fundamental theorem of arithmetic and a few other properties of primes which I am assuming anyway. I might as well assume that this fact is true. There was another fact that I assumed was true that uses some properties of integers that I don't think that we will get into.

**Lemma 2.** *For positive integers $c, d, e$,*

$$gcd(d, e) = 1 \text{ if and only if } gcd(cd, ce) = c$$

Take again the example of $gcd(10, 12) = 2$ and compare this to $gcd(5, 6) = 1$.

Now I claim that I have enough information to write down the formula for the number of necklaces with $n$ beads using $k$ colors and this formula is written in terms of a quantity $\phi(d)$ = the number of integers $e$ between 1 and $d$ that are relatively prime to $d$.

$$\#\text{necklaces with } n \text{ beads colored with } k \text{ colors} = \frac{1}{n}\sum_{d|n}\phi(d)k^{n/d} \tag{1}$$

I am now thinking about it and I am not sure I mentioned why this is even useful. If you don't know a formula for $\phi(d)$, then we have given one formula that is hard to compute (Burnside's lemma) in terms of another (the formula in equation (1) in terms of $\phi(d)$). The thing is that there are formulas for $\phi(d)$. If $d$ has a factorization into distinct primes $p_1^{a_1}p_2^{a_2}\cdots p_\ell^{a_\ell}$ then

$$\phi(d) = (p_1^{a_1} - p_1^{a_1-1})(p_2^{a_2} - p_2^{a_2-1})\cdots(p_\ell^{a_\ell} - p_1^{a_\ell-1}) \ .$$

For example $\phi(8) = 2^3 - 2^2 = 8 - 4 = 4$. But this is a side note.

There are $n$ group elements which act on this necklace $R_1, R_2, R_3, \ldots, R_n = R_0 = e$. We have already deduced that $R_r$ consists of cycles of length $d$ if and only if $lcm(r,n) = rd$ and since $lcm(r,n) = rn/gcd(n,r)$ then it must be that the length of the cycle is $d = n/gcd(n,r)$ (verify that this actually happens on an example) and so $gcd(n,r) = n/d$.

But because of Lemma 2 above, we have that $gcd(n,r) = n/d$ if and only if $gcd(d, rd/n) = 1$. This means that for every $e = rd/n$ which is relatively prime to 1, there is an $r = \frac{n}{d}e$. This says that there is a bijection between the set $\Phi(d) = \{e : gcd(d,e) = 1\}$ and the set $\Psi(d) = \{r : gcd(n,r) = n/d\}$, and moreover the bijection from $\Phi(d)$ to $\Psi(d)$ is to multiply the elements of $\Phi(d)$ by $n/d$.

Therefore we know that there are $\phi(d) = |\Phi(d)|$ elements with $n/d$ cycles of length $d$ and so there are $k^{n/d}$ ways of coloring each of those $n/d$ cycles. Burnside's Lemma then says that

$$\#\text{ necklaces} = \frac{1}{n}\sum_{r=1}^{n} Fix(R_r) = \frac{1}{n}\sum_{r=1}^{n} k^{gcd(n,r)} = \frac{1}{n}\sum_{d|n}\phi(d)k^{n/d} \ .$$

Recall that for our example of $n = 8$, we had the table of

| $g \in G$ | cycle notation |
|---|---|
| $R_0 = R_8$ | (1)(2)(3)(4)(5)(6)(7)(8) |
| $R_1$ | (18765432) |
| $R_2$ | (1753)(2864) |
| $R_3$ | (16385274) |
| $R_4$ | (15)(26)(37)(48) |
| $R_5$ | (147258361) |
| $R_6$ | (1357)(2468) |
| $R_7$ | (12345678) |

And when we grouped them by the elements that consist of $n/d$ cycles of length $d$. Then the following table agrees with this construction.

| $d=$ cycle length | integers between 1 and $d$ that are relatively prime to $d$ | motions which have $n/d$ cycles of length $d$ |
|---|---|---|
| 8 | $\{1,3,5,7\}$ | $\{R_1,R_3,R_5,R_7\}$ |
| 4 | $\{1,3\}$ | $\{R_2,R_6\}$ |
| 2 | $\{1\}$ | $\{R_4\}$ |
| 1 | $\{1\}$ | $\{R_8\}$ |

For this example the ways of coloring a necklace with 8 beads and $k$ colors is equal to

$$\frac{1}{8}(k^8+k^4+2k^2+4k)$$

We can also apply Polya's theorem to get a refinement of this formula. Since the generating function for the ways of coloring a single cycle of length $d$ is equal to $\sum_{i=1}^k x_i^d$, then by the multiplication principle of generating functions, the generating function for the number of ways of coloring $n/d$ cycles of length $d$ is equal to $\left(\sum_{i=1}^k x_i^d\right)^{n/d}$. Moreover, Polya's Theorem says that the generating function for the number of ways of coloring the necklaces with $k$ colored beads will be

$$\frac{1}{n}\sum_{d|n}\phi(d)\left(\sum_{i=1}^k x_i^d\right)^{n/d} .$$

Lets try this in practice for $n=8$, the generating function will be

$$\frac{1}{8}((R+B)^8+(R^2+B^2)^4+2(R^4+B^4)^2+4(R^8+B^8))$$

Lets expand this with Sage (although I also did it by hand for a single coefficient):

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sage: ( (R+B)^8 + (R^2+B^2)^4 + 2*(R^4 + B^4)^2 + 4*(R^8+B^8))/8
1/8*(B + R)^8 + 1/8*(B^2 + R^2)^4 + 1/4*(B^4 + R^4)^2 + 1/2*B^8 + 1/2*R^8
sage: expand(_)
B^8 + B^7*R + 4*B^6*R^2 + 7*B^5*R^3 + 10*B^4*R^4 + 7*B^3*R^5 + 4*B^2*R^6
 + B*R^7 + R^8
```

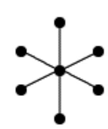
What this says is that there are 7 necklaces with 5 blue beads and 3 red beads, they are

$$BBBBBrrr,\ BBBBrBrr,\ BBBrBBrr,\ BBrBBBrr,$$
$$BrBBBBrr,\ BBBrBrBr,\ BBrBBrBr$$

Check very carefully and I THINK that all 7 of these are different and if they are, then every necklace is equivalent to one of these.

Next time I want you to work on the combinatorics problem that I posed last time:

How many colorings of the graph



are there using $k$ colors such that each color is used at most twice?