NOTES ON OCT 16, 2012

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I started off by giving an example that was typical of the type of problem that I have been giving in the homework and the midterm. I felt that at this point you should be prepared to this type of problem:

How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = n$$

with $x_1 + x_2$ divisible by 3?

The first step would be to find an equation for the generating function, although there is a second answer (that I didn't discuss in class) which also can be used to answer this question. To find the generating function you would first break the problem into three steps, find the number of solutions to $x_1 + x_2 = k$ where $x_1 + x_2$ is divisible by 3, the number of solutions to $x_3 = \ell$ and the number of solutions to $x_4 = n - k - \ell$. Since we have broken down the problem into these three substeps, then we know that

g.f. for # of solutions to $x_1 + x_2 + x_3 + x_4 = n$ with $x_1 + x_2$ divisible by 3 = (g.f. for # of solutions to $x_1 + x_2 = n$ with $x_1 + x_2$ divisible by 3) (g.f. for # of solutions to $x_3 = n$) (g.f. for # of solutions to $x_4 = n$). We know that for each n there is one solution to $x_3 = n$ so the generating function for the number of such solutions is $1/(1-x)$ (similarly for $x_4 = n$). Now to find the generating function for $x_1 + x_2 = n$ with $x_1 + x_2$ divisible by 3, the obvious way is to take the generating function for the number of solutions to $x_1 + x_2 = n$ which we know is equal to $A(x) = 1/(1-x)^2$ and then pick out every third term using the method that we discussed on October 2 (see notes) and give it was

$$\frac{1}{3}(A(x) + A(\zeta_3 x) + A(\zeta_3^2 x))$$

Instead I will suggest another method to get at this generating function by computing a table of coefficients and then writing a formula for the generating function.

n	0	1	2	3	4	5	6	7	8	9
# of solutions	1	0	0	4	0	0	7	0	0	10

In other words if n is divisible by 3, then the number of solutions is $n+1$, otherwise it is 0. This means that the generating function is

$$\sum_{n\geq 0}(3n+1)x^{3n} = 3\sum_{n\geq 0} n x^{3n} + \sum_{n\geq 0} x^{3n} = 3\frac{x^3}{(1-x^3)^2} + \frac{1}{1-x^3}\ .$$

Note that the last equality comes from the tables of generating functions that we have developed. From this we conclude that

g.f. for # of solutions to $x_1 + x_2 + x_3 + x_4 = n$ with $x_1 + x_2$ divisible by 3 =

$$\left(3\frac{x^3}{(1-x^3)^2} + \frac{1}{1-x^3}\right)\frac{1}{(1-x)^2}\ .$$

There is another way of coming up with an answer to this question. If we want to find the number of solutions to $x_1 + x_2 + x_3 + x_4 = n$ with $x_1 + x_2$ divisible by 3 then we don't need to go so far as to apply generating functions. This is equal to the number of solutions to $x_1 + x_2 = k$ and $x_3 + x_4 = n - k$ with $x_1 + x_2$ divisible by 3. For each k there are $(k+1)$ solutions to $x_1 + x_2 = k$ and there are $n - k + 1$ solutions to $x_3 + x_4 = n - k$. Therefore the number of solutions

$$\sum_{3\ divides\ k} (k+1)(n-k+1) = \sum_{r=0}^{\lfloor n/3 \rfloor} (3r+1)(n-3r+1).$$

What I hoped to show from this example is that ordinary generating functions are a very powerful tool for enumerating certain types of sets. Usually these are sets that can be reduced to something that is very similar to the example we just looked at. We can apply the multiplication principle of generating functions if we can divide the enumeration of a set with c_n elements into a widget of size k and a doodle of size $n-k$, then if a_k is the number of widgets of size k and b_{n-k} is the number of doodles of size $n-k$, then

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}\ . \tag{1}$$

The problem is that there are many other enumeration questions where we don't have this sort of decomposition. One of those examples are the Bell numbers: $B_0 = 1$, $B_1 = 1$ and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ for $n > 1$ which is equal to the number of set partitions of $n+1$. We can calculate the next few values as $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$.

The problem is that the expression $\sum_{k=0}^{n} \binom{n}{k} B_k$ is not of the form $\sum_{k=0}^{n} a_k b_{n-k}$. Why? If I set $B(x) = \sum_{n \geq 0} B_n x^n$, then when I multiply $B(x)A(x)|_{x^n}$ is $\sum_{k=0}^{n} a_{n-k} B_k$ and I can't find a generating function where $a_{n-k} = \binom{n}{k}$. It just doesn't seem to work.

There is a way around this. We can define a new type of generating function $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$ and if we take a second to $B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$ and multiply these together then we see that

$$A(x)B(x) = \sum_{n \geq 0} \frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!} x^n = \sum_{n \geq 0} \frac{n!}{k!(n-k)!} a_k b_{n-k} \frac{x^n}{n!} = \sum_{n \geq 0} \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!}$$

.

This gives us a new principle to work with.

Principle 1. *The coefficient of $x^n/n!$ in the product of $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ is equal to*

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}. \tag{2}$$

I mention this because in the recurrence for B_{n+1} if we set $a_k = B_k$ and $b_{n-k} = 1$ then it is of this form. Therefore it seems as though we might be able to write down a generating function of this form. We call $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ the exponential generating function for a sequence. Consider the exponential generating function for the sequence $1, 1, 1, 1, 1, 1, \ldots$,

$$\sum_{n\geq 0} 1 \frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^n}{n!} = e^x \ .$$

The exponential generating function for the sequence $0, 1, 2, 3, 4, 5, 6, \ldots$, is equal to

$$\sum_{n\geq 0} n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{(n-1)!} = xe^x \ .$$

Now consider the sequence $\binom{0}{k}, \binom{1}{k}, \binom{2}{k}, \binom{3}{k}, \binom{4}{k}, \binom{5}{k}, \ldots$, where k is fixed. We calculate that the exponential generating function is equal to

$$\sum_{n\geq 0} \binom{n}{k} \frac{x^n}{n!} = \sum_{n\geq k} \frac{n!}{k!(n-k)!} \frac{x^n}{n!} = \sum_{n\geq k} \frac{1}{k!} \frac{x^n}{(n-k)!} = \frac{x^k}{k!} \sum_{n\geq k} \frac{x^{n-k}}{(n-k)!} = \frac{x^k}{k!} e^x \ .$$

Now lets apply what we know to finding a formula for the exponential generating function for $B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$ where $B_0 = B_1 = 1$ and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Lets work it out as we normally do except with exponential generating functions.

$$\begin{aligned}
B(x) &= \sum_{n\geq 0} B_n \frac{x^n}{n!} \\
&= 1 + \sum_{n\geq 1} B_n \frac{x^n}{n!} \\
&= 1 + \sum_{n\geq 1} \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \frac{x^n}{n!} \\
&= 1 + B_0 \frac{x}{1!} + \left(\binom{1}{0} B_0 + \binom{1}{1} B_1 \right) \frac{x^2}{2!} + \left(\binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 \right) \frac{x^3}{3!} + \cdots
\end{aligned}$$

Now those coefficients that are appearing in this sum should look very familiar. They are exactly those that appear in equation (2) except that $a_k = B_k$ and $b_{n-k} = 1$. Therefore if we calculate $B(x)e^x$ we see

$$B(x)e^x = B_0 + \left(\binom{1}{0} B_0 + \binom{1}{1} B_1 \right) \frac{x^1}{1!} + \left(\binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 \right) \frac{x^2}{2!} + \cdots$$

We can make the expression for $B(x)e^x$ look exactly like the expression that comes after the 1+ in the expression for $B(x)$ by integrating one time. What this means is that

$$B(x) = 1 + \int B(x)e^x dx$$

or also

$$B'(x) = B(x)e^x$$

It is not trivial to solve for $B(x)$, but it is possible and if I give you the solution, it is not hard to verify that $B(x) = e^{e^x-1}$. In fact if I use "sage" to compute the Taylor expansion of e^{e^x-1}, then I see that

```
sage: taylor(exp(exp(x)-1), x, 0, 6)
203/720*x^6 + 13/30*x^5 + 5/8*x^4 + 5/6*x^3 + x^2 + x + 1
```

If I rewrite this with the $n!$ in the dominators (no simplification of the fractions) then I see that

$$e^{e^x-1} = 1 + \frac{x}{1!} + 2\frac{x^2}{2!} + 5\frac{x^3}{3!} + 15\frac{x^4}{4!} + 52\frac{x^5}{5!} + 203\frac{x^6}{6!} + \cdots$$

and this agrees with what we calculated earlier with B_0 through B_5.

I can also use sage to help me with the algebra of verifying that $B'(x) = \frac{d}{dx}(e^{e^x-1}) = e^{e^x-1}e^x = B(x)e^x$.

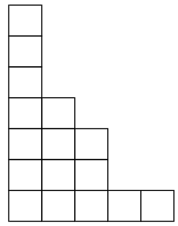
```
sage: diff(exp(exp(x)-1),x)
e^(x + e^x - 1)
sage: exp(x)*exp(exp(x)-1)
e^(x + e^x - 1)
```

I will continue to expand on the use of exponential generating functions. What we will need to do is develop tools for creating libraries of generating functions as we did for ordinary generating functions. For instance, if I give you the exponential generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$, then I expect you to be able to give me expressions for $\sum_{n\geq 0} a_{n+2}\frac{x^n}{n!}$, $\sum_{n\geq 0} na_n\frac{x^n}{n!}$, $\sum_{n\geq 0} a_n\frac{x^{n+2}}{(n+2)!}$.

I would also like to apply our generating function techniques to objects called partitions because they are very much the type of combinatorial object where equation (1) applies, just as the recursion for the number of set partitions B_n was able to use (2).

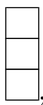
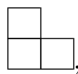
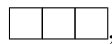
Recall that a *partition* of n is a sum $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. The order of the sum doesn't matter so to avoid confusion we assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. The λ_i are called the *parts* of the partition. r here is the number of parts of the partition or the *length* of the partition. The *sizes* of the parts are the values λ_i. The *size* of the partition is the sum of the sizes of all the parts (in this case n). Parts are called *distinct* if they are not equal to each other. The *number of parts of a given size* refers to the number of times that a value appears as a part.

A partition is represented by a diagram where I put rows of boxes and in the i^{th} row from the the bottom I put λ_i boxes and these rows of boxes are left justified. For instance the partition $(5, 3, 3, 2, 1, 1, 1)$ is represented as

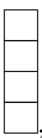
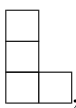
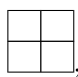
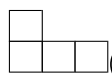
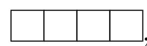


The picture is a convenient way of picturing what a partition is as a combinatorial object. Here are some examples:

Partitions of 3

, $(1, 1, 1)$ , $(2, 1)$ , (3)

Partitions of 4

, $(1, 1, 1, 1)$ , $(2, 1, 1)$ , $(2, 2)$ $(3, 1)$ , (4)

Lets first consider the generating function for partitions using parts of size k only. Define $\mathcal{P}_{=k}(x) = \sum_{n\geq 0}(\text{number of partitions of } n \text{ with parts of size equal to } k)x^n$. The only partitions of this type are the empty partition $()$, (k), (k, k), $(k, k, k), \ldots$. There is exactly one partition of n with parts of size k iff k divides n. Therefore the generating function is simply

$$\mathcal{P}_{=k}(x) = 1 + x^k + x^{2k} + x^{3k} + \cdots = \frac{1}{1 - x^k}.$$