HOMEWORK #1 - MATH 4160

ASSIGNED: SEPT 20, 2016 DUE: THURSDAY OCT 2, 2014

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers.

Stirling numbers and

(1) In class, we defined for $k > 0$, $(x)_k := x(x-1)(x-2)\cdots(x-k+1)$ (falling factorial) and $(x)^{(k)} := x(x+1)(x+2)\cdots(x+k-1)$ (rising factorial). Both $(x)_k$ and $(x)^{(k)}$ are products of k terms. Prove the following by two means, by induction and by telescoping sums: For $k \geq 1$ and $n \geq 0$,

$$\sum_{i=1}^{n} (i)_{k+1} = (1)_{k+1} + (2)_{k+1} + \cdots + (n)_{k+1} = (n+1)_{k+2}/(k+2) \ .$$

(2) For $n \geq 0$ and $1 \leq k \leq n$, let $s'(n,k) =$ the number of permutations of $[n]$ that have k disjoint cycles (these are called the signless Stirling numbers of the first kind). Explain why in 2-3 sentences that

$$s'(n,k) = (n-1)s'(n-1,k) + s'(n-1,k-1) \ .$$

Use this recursion to compute the values of $s'(n,k)$ for $1 \leq n \leq 5$ and $1 \leq k \leq n$. Show explicitly that $s'(4,2) = 11$ by listing the 11 permutations of $\{1,2,3,4\}$ into two cycles.

(3) Show that

$$(x)_n = \sum_{k=1}^{n} (-1)^{n-k} s'(n,k) x^k \ .$$

Enumeration problems. I am giving you the answer so the emphasis is on you coming up with a clear explanation of why the answer is true. I don't care what the answer is, I care why the answer is....

(1) How many 6 card hands have a pair (a 2 of a kind) so that no better poker pattern appears? A is either high or low, but not both. Answer: 9,730,740

(2) An urn contains seven red balls and eight black balls and six blue balls. Three balls are randomly chosen from the urn. Find the probability that all three balls are the same color. Answer: ≈ 0.08346

(3) How many straight poker hands are there (not straight flush) from a deck of 50 cards because it has the five and nine of spades missing? A is high or low. Answer: 7,713

Combinatorial proofs: For the following problems give a combinatorial proof by describing a set that is counted by the left hand side of the equality and a set that is counted by the right hand side of the equality and explaining why these two sets are the same.

(1)
$$(n)^{(3)} = 2n + 3n^2 + n^3$$
(note: depending on how you explain what $(n)^{(3)}$ represents, the right hand side might be easier to explain as $2n + 2n^2 + n^2 + n^3$).

(2)
$$\binom{2n+1}{2} = 2\binom{n+1}{2} + n^2$$

(3)
$$\binom{2n}{n}\binom{2n}{n+1} = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{n-k}\binom{n}{n+1-k}$$