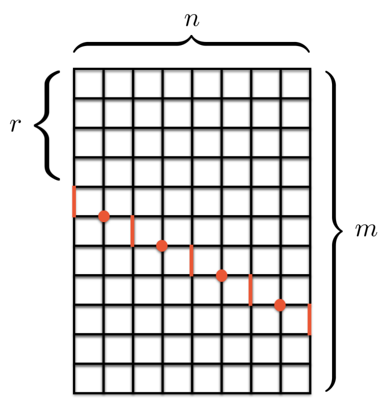
HOMEWORK #2 - MATH 4160

ASSIGNED: OCT 9, 2014 DUE: OCT 21, 2014

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers.

(1) Use the following diagram to arrive at a binomial formula involving paths in an $n \times m$ rectangle from the point $(0,0)$ to (n,m) using only steps in the $(1,0)$ or $(0,1)$ direction where each term in the sum is the number of paths which pass through either one of the red vertical segments or dots. Your identity should involve three unknowns m, n, r and should be explained as generally as possible. Show that your identity holds in two cases $m = 4$,$n = 3$,$r = 0$ and $m = 4$,$n = 4$,$r = 2$.



(2) Find a formula for the generating function

$$(1)^{(k)} + (2)^{(k)}x + (3)^{(k)}x^2 + (4)^{(k)}x^3 + (5)^{(k)}x^4 + \cdots = \sum_{n \geq 0} (n+1)^{(k)} x^n$$

(3) Use the last result and the formula $n^r = \sum_{k=1}^{r} (-1)^{r-k} S(r,k)(n)^{(k)}$ to find a formula for the generating function

$$\sum_{n \geq 0} (n+1)^r x^n$$

Use your formula to verify in particular that

$$\sum_{n \geq 0} (n+1)x^n = \frac{1}{(1-x)^2}, \qquad \sum_{n \geq 0} (n+1)^2 x^n = \frac{1+x}{(1-x)^3}, \qquad \sum_{n \geq 0} (n+1)^3 x^n = \frac{1+4x+x^2}{(1-x)^4},$$

(4) Find the generating function for the ways of distributing n dollars in loonies and twonies to 5 people satisfying the following conditions:
  (a) no restriction
  (b) the first two people have an even number of twonies between them
  (c) the first two people have at most 6 twonies together

(d) the first two people have together at most 6 twonies together and an even number

(e) the first two people have together either at most 6 twonies together or an even number between them

Check your answer on the computer to verify your answer that if there are 20 dollars all together to distribute to 5 people then for condition (a) there are 700,128 ways; for condition (b) there are 383,528 ways; for condition (c) there are 694,688 ways; for condition (d) there are 382,428 ways; for condition (e) there are 695,788 ways.

(5) A 7 digit phone number does not begin with a 0,1 or a 9, otherwise there are no other restrictions. How many 7 digit phone numbers are there whose sums of digits is 32? Answer: 373,024.

(6) Give a formula for the generating function $P(x) = \sum_{n\geq 0} \binom{2n+1}{2} x^n$, $Q(x) = \sum_{n\geq 0} \binom{n+1}{2} x^n$ and $R(x) = \sum_{n\geq 0} n^2 x^n$. Show that $P(x) = 2Q(x) + R(x)$ and this implies that the coefficient of x^n in the LHS is equal to the coefficient of x^n in the right hand side and hence

$$\binom{2n+1}{2} = 2\binom{n+1}{2} + n^2 \ .$$

(7) Let $B_k(x) = \sum_{n\geq 0} \binom{n-1}{k-1} x^n$. Show that $B_k(x)B_\ell(x) = B_{k+\ell}(x)$. Take the coefficient of x^n in both sides of this equation and explain what identity this shows about binomial coefficients.

(8) The Lucas numbers are the sequence defined by $L_1 = 1$, $L_2 = 3$ and $L_{n+1} = L_n + L_{n-1}$. The Lucas generating function is $L(q) = \sum_{n\geq 0} L_{n+1} q^n = \frac{1+2q}{1-q-q^2}$. Compute a formula for $L_{odd}(q) = \sum_{n\geq 0} L_{2n+1} q^n$ and $L_{even}(q) = \sum_{n\geq 0} L_{2n+2} q^n$. Recall also that we showed in class that $F_{even}(q) = \sum_{n\geq 0} F_{2n+2} q^n = \frac{1}{1-3q+q^2}$. Show that $L_{odd}(q)\frac{1}{1+q} = F_{even}(q)$. Derive from this an identity relating F_{2n+2} as an alternating sum of odd Lucas numbers. Show that the identity is true for $n = 0, 1, 2, 3$.