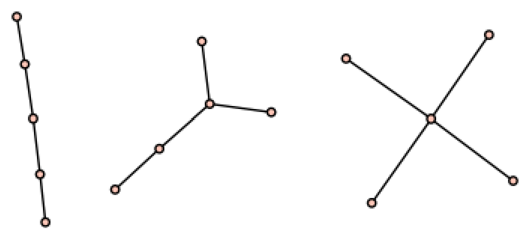
HOMEWORK #4 - MATH 4160

ASSIGNED: MONDAY NOVEMBER 17, 2014 DUE: NOVEMBER 27, 2014

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers.

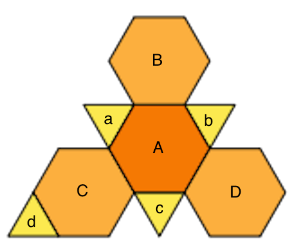
(1) For each of the following three trees (they are called trees because there are no loops and only branches), how many different ways are there of coloring the vertices with k colors so that no two adjacent vertices are colored with the same color?

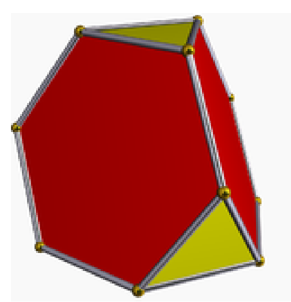


(2) For each of the trees in the first problem, how many different ways are there of coloring the vertices with k colors where two colorings of the graph are considered different if there is no rearrangement of the vertices so that the colorings look the same?

(3) For each of the trees in the first problem, how many different ways are there of coloring the vertices with k colors such that adjacent vertices are colored with different colors and so that two colorings of the graph are considered different if there is no rearrangement of the vertices so that they look the same?

(4) The truncated tetrahedron (`http://en.wikipedia.org/wiki/Truncated_tetrahedron`) is one of the 13 Archimedean solids. It has 4 hexagonal faces and 4 triangular faces.





Allowing for only rotations of the object (no mirror reflections), how many elements are there in the group of symmetries of this object? Explain clearly your count. List the elements of the group as a permutation of the 8 faces.

(5) How many different (under motions of the symmetry group) ways are there of coloring the faces of the truncated tetrahedron with black and white?
(6) How many ways are there of coloring the faces of the truncated tetrahedron with black and white using four black faces and four white faces?
(7) How many ways are there of coloring the faces of the truncated tetrahedron with black and white using 2 black triangular faces, 2 white triangular faces, 2 red hexagonal faces and 2 green hexagonal faces?

Generating functions

(1) Recall that the (unsigned) Stirling numbers of the first kind were satisfy the recursive formula $s'(n,0) = 0$ and $s(0,n) = 0$ for $n > 0$, $s'(0,0) = 1$ and for $n \geq 1$ and $1 \leq k \leq n$,
$$s'(n,k) = (n-1)s'(n-1,k) + s'(n-1,k-1).$$
Use this recursion to show that
$$e^{-ulog(1-x)} = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} s'(n,k) u^k \frac{x^n}{n!} \ .$$
(2) Use the last result to find a formula for the number permutations of n into a odd number of cycles. Hint: the number of permutations of 10 in an odd number of cycles is equal to 1814400.
(3) (bonus) Recall in the last homework assignment I asked you to give a generating function for the number of set partitions into an odd number of parts. You should have found that if
$$B_o(n) = \sum_{k \geq 0} S(n, 2k+1)$$
then
$$\mathcal{B}_o(x) = \sum_{n \geq 0} B_o(n) \frac{x^n}{n!} = \frac{e^{e^x - 1} - e^{-(e^x-1)}}{2} \ .$$
Use this result to show that
$$B_o(n) = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{r=0}^{2m+1} \frac{(-1)^{2m+1-r} r^n}{r!(2m+1-r)!}$$
Show that this formula works in the case of $n = 1, 2, 3, 4$ by using the formula to calculate $B_o(1), B_o(2), B_o(3), B_o(4)$ and listing all of the set partitions of size ≤ 4 into an odd number of parts and showing that they agree.