NOTES FROM THE FIRST CLASS

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The course web page and description is at

`http://garsia.math.yorku.ca/ zabrocki/math4160f14/`

The first thing I did was try to explain what combinatorics is about and what we will learn in this class. The main takeaway message is "combinatorics = counting" and it has applications through all types of mathematics. My research is in the area of algebraic combinatorics where I use the techniques of studying discrete structures in algebraic constructions such as modules, algebras, groups and rings. These sorts of questions lead to beautiful mathematics.

I mentioned that one of the things that is great about combinatorics as a subject is that it is "elementary." That is, that it is based on very little background knowledge of other mathematics. The word elementary in mathematics does *not* mean that the subject is easy, just that it is possible to understand the mathematics from very few principles.

The course will consist of homework assignments and take home exams. I would encourage you to work together on the homework assignments, but I want you to hand in your own work (and not just copying). FYI, the last time I taught this class two students went to see the associate dean over concerns of cheating. Once I feel that there is an issue it will be for their office to resolve.

The take home exams will be similar to the homework. On these I will be explicit that I expect you to work alone. You can come and ask me questions and I will help as best as I can, but I don't want you to discuss with your classmates or online resources.

I started then talking about combinatorics and about three tools that we will use.

1. *The equality principle* - If there exists a bijection between two sets A and B then $|A| = |B|$ (note that $|A|$ is the symbol I will use for the number of elements in the set A).

Example: Consider "the set of subsets of $\{1, 2, \ldots, n\}$ that contain both 1 and n" and "the set of subsets of $\{1, 2, \ldots, n-2\}$. I claim that there is a bijection between both of these two sets because if we take $\{1, n, a_1, a_2, \ldots, a_k\}$ as a set which contains both 1 and n and the $2 \leq a_i \leq n-1$ then this is sent to set $\{a_1 - 1, a_2 - 1, \ldots, a_k - 1\}$ is a subset of $\{1, 2, \ldots, n-2\}$.

In the case of $n = 4$ we see that there are four subsets which contain $\{1, 4\}$, namely $\{\{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}$. There are also four subsets of $\{1, 2\}$, namely $\{\{\}$, $\{1\}$, $\{2\}$, $\{1, 2\}\}$.

I will often not state that I am using this principle, just that one description is equal to another and then so "clearly" these two sets have the same number of elements are equal. The word *clearly* is loaded. What it means is that there is some detail to be understood at this point and you need to figure it out (and hence it probably isn't clear in any way).

2. *The addition principle* - If there are three sets related by $A = B \uplus C$ (which means A is the union of B and C and both B and C don't have elements in common), then $|A| = |B| + |C|$.

Example: Lets set $\binom{n}{k}$ to be a symbol which represents the number of subsets of an n element set with exactly k elements. So, for instance $\binom{4}{2} = |\{\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\}\}|$ Consider $n \geq k \geq 1$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof: Every k element subset of the set $\{1, 2, \ldots, n\}$ either contains n or it does not. If the set contains n, then it has $k-1$ other elements from $\{1, 2, \ldots, n-1\}$. Otherwise, the set is has k elements from $\{1, 2, \ldots, n-1\}$.

3. *The multiplication principle* - If the set A consists of all pairs (x, y) where x is an element of B and y is an element of C, then $|A| = |B| \cdot |C|$.

Example: Again consider the case when $n \geq k \geq 1$, then

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$$

Proof: The left hand side of this equation represents the number of pairs consisting of a subset of $\{1, 2, \ldots, n\}$ of with k elements and a choice of one of the k elements which will be colored orange.

The right hand side consists of all pairs whose first element is one of the numbers a where $1 \leq a \leq n$ to painted orange, followed by a $k-1$ element subset of $\{1, 2, \ldots, n\} \backslash \{a\}$.