# FOURTH LECTURE : SEPTEMBER 18, 2014

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I started off by listing the building block numbers that we have already seen and their combinatorial interpretations.

- $S(n,k) =$ the number of set partitions of $\{1,2,\ldots,n\}$ into $k$ parts
- $B(n) =$ the number of set partitions of $\{1,2,\ldots,n\}$
- $s'(n,k)$ the number of permutations of $\{1,2,\ldots,n\}$ that have $k$-cyles
- $P(n) = n! =$ the number of permutations of $\{1,2,\ldots,n\}$
- $\binom{n}{k} =$ the number of subsets of $\{1,2,\ldots,n\}$ that contain $k$ elements

Then we still need combinatorial explanations for $n^k$, $(n)_k$ and $(n)^{(k)}$. You can be super creative when you do this or extremely boring and express it in terms of sets and lists.

After a little discussion I mentioned that we can imagine that we have $n$ colors of paints and $k$ ordered objects then by the multiplication principle if we pick one of the $n$ colors of paints for the first one, $n$ for the second, $n$ for the third, etc. then the total number of ways of coloring those $k$ ordered objects is $n^k$.

You can and should be more creative than I was. Imagine that you have you have $k$ people at dinner and they each order one of $n$ desserts. Since each person has $n$ choices for the dessert then there are $n^k$ possible ways that the desserts can be ordered.

The standard answer is

$n^k =$ the number of sequences of length $k$ whose entries are $\{1,2,\ldots,n\}$

I then noted that since $(n)^{(k)} = n(n+1)(n+2)\cdots(n+k-1) = (n+k-1)_k$ that we just need to come up with a combinatorial interpretation for $(n)_k$. It turns out that this was a mistake. We should have come up with a combinatorial interpretation for $(n)^{(k)}$ separately.

Let me tell you what the standard interpretations are and then I expect you to give a 1-2 line explanation of why that is the case.

$(n)^{(k)} =$ the number of sequences of length $k$ whose $i^{th}$ entry is between 1 and $n+i-1$ for $1 \le i \le k$

$(n)_k =$ the number of sequences of length $k$ whose entries are in $\{1,2,\ldots,n\}$ where each entry is different

Then I stupidly tried to use the explanation of

$$(2)^{(3)} = 2(2+1)(2+2)24 = 2 \cdot 2^1 + 3 \cdot 2^2 + 2^3$$

(this is a specific case of the more general formula $(n)^{(3)} = 2 \cdot n^1 + 3 \cdot n^2 + n^3$ that I want you to explain for homework). Where I went wrong is that I need to to use the combinatorial interpretation for $(2)^{(3)}$ and not the one for the falling factorial (they both can be done, but using the interpretation for $(4)_3$ is less clear and more tricky. Let me show it here.

The number $(2)^{(3)} = 24$ is the number of sequences whose first entry is between 1 and 2, whose second entry is between 1 and 3 and whose third entry is between 1 and 4. The 24 sequences are

$$111, 112, 113, 114, 121, 122, 123, 124, 131, 132, 133, 134$$

$$211, 212, 213, 214, 221, 222, 223, 224, 231, 232, 233, 234$$

Now if we rewrite this an express emphasize where the $2^1$, $2^2$ and $2^3$ are coming from then we see $(2)^{(3)} = \mathbf{2}(\mathbf{2}+1)(\mathbf{2}+2) = 2 \cdot \mathbf{2} + 2\mathbf{2}^2 + \mathbf{2}^2 + \mathbf{2}^3$.

We will take a bold face **2** to mean that the entry in the sequence is either a 1 or a 2 and the other numbers which appear to mean that it is a 3 or a 4.

- $2 \cdot \mathbf{2}$ is equal to the number of sequences whose last two entries are either a 3 or a 4 (they are 133, 134, 233 and 234)
- $2\mathbf{2}^2$ is equal to the number of sequences whose first two entries are 1 or 2 and the last one is either 3 or 4 (they are $113, 114, 123, 124, 213, 214, 223, 224$)
- $\mathbf{2}^2$ is equal to the number of sequences whose first and third entries are 1 or 2 and the middle one is 3 (they are $131, 132, 231, 232$)
- $\mathbf{2}^3$ represents the number of sequences all of whose entries are 1 or 2 (they are $111, 112, 121, 211, 122, 212, 221, 222$)

Since it is always the case that the 3 or 4 appear in the last two positions, every sequence in the interpretation of $(2)^{(3)}$ falls into one of these 4 categories and hence they must also sum to 24 and as a by the addition principle, $(2)^{(3)} = 2 \cdot 2^1 + 3 \cdot 2^2 + 2^3$.

But since I messed up that explanation I skipped to counting hands of cards. Poker is a card game played with a 52 card deck with 13 values for the cards and 4 suits. Poker hands are ranked by how common a hand is.

The types of poker hands are:

- straight flush : a sequence of 5 cards in order all with the same suit (there is also a royal flush, $10, J, Q, K, A$ all the same suit, but these are also straight flushes so there is no real reason to separate them)
- 4 of a kind - 4 cards of the same value and one extra card
- full house - a pair and a three of a kind
- flush - five cards all one suit not a straight

- straight - five cards whose values are in a 5 card sequence and it is not the case they all have the same suit
- 3 of a kind - three cards of the same value and two extra cards with different values
- two pairs
- pair
- none of the above (often called 'high card')

A really good exercise is to figure out a way of counting the number of each of these sets using only addition (like I said, this is sometimes the more complicated way of coming up with the answer) and then add them all up and check that they add up to $\binom{52}{5}$ (the number of ways of picking 5 cards from a deck of 52).

The categories that we describe as a 'poker hand' are not the only descriptions possible, because I could make up a description and call that a poker hand. For example a 'sandwich two pair' is a 5 card hand that contains two pairs such that the two pairs have the same two suits and the 5th card has a value which is between the two pairs and has a suit which is the same as the pairs (e.g. $4\heartsuit, 4\clubsuit, 9\heartsuit, 9\clubsuit, 6\clubsuit$ is an example of a sandwich two pair, but $4\heartsuit, 4\clubsuit, 9\heartsuit, 9\clubsuit, J\clubsuit$ and $4\heartsuit, 4\clubsuit, 9\heartsuit, 9\spadesuit, 6\heartsuit$ are not).

Here is how you count all of the numbers of different types of hands.

**straight flush:**
There are also 40 straight flush hands because there are 4 possible suits and 10 possible straights (that begin with $A$ through 10 as the lowest card, the A is either a high or low card but not both).

**4-of-a-kind:**
For instance, there are $13 \times 48 = 624$ possible 4-of-a-kind hands because we can choose which value appears 4 times in a 4-of-a-kind hand plus one extra card from the remaining 48 cards in the deck. Therefore a straight flush beats a 4-of-a-kind because there are more 4-of-a-kind hands than straight flush.

**full house:**
A full house consists of a three-of-a-kind and a pair. To specify one of these hands we must know the value of the three-of-a-kind, the three suits which appear in the three-of-a-kind, the value of the pair, and the two suits which appear in the pair. There are 13 values for the three-of-a-kind, $\binom{4}{3}$ ways of specifying the suits, 12 values for the pair and $\binom{4}{2}$ ways of specifying the suits. By the multiplication principle there are $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3,744$.

**flush:**
A flush hand has 5 values which are all different and one of the four suits. Now if you pick the 5 values from the 13 possible you will still include those sequences that are straight

flushes, so there are $\binom{13}{5} - 10$ possible sets of values and 4 suits. In total there are $\left(\binom{13}{5} - 10\right) \cdot 4 = 5,108$ possible flush hands

**straight:**
If a hand is a straight but not a straight flush then there are 10 possible straights and there are $4^5$ ways of picking a suit for each of the cards of the straight, BUT we have to subtract off the number of ways that all suits are the same. By the multiplication principle we know that there are $10 \cdot (4^5 - 4) = 10,200$ possible straight hands which don't have a flush.

**3-of-a-kind:**
This hand is determined by which value is repeated three times, the three suits that appear and then two other cards from the remaining 48 (because we remove all the cards that are of the same value as the 3-of-a-kind) which do not form a pair. Since there are $12 \cdot \binom{4}{2}$ possible ways of making a pair from the remaining 48 cards, there are $\left(\binom{48}{2} - 12 \cdot \binom{4}{2}\right) = 1,056$ two cards which do not form a pair. Alternatively, we can pick two values from the remaining 12 and then a suit for each of those cards so there are also $\binom{12}{2} \cdot 4^2 = 1,056$ ways of picking the pair. In total there are $13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2 = 54,912$ three of a kind hands.

**2-pair:**
I also counted the number of hands with exactly two pairs. The following information completely determines a hand that has a two pair.

- two values (an upper and a lower) which will each appear twice in the hand
- two suits of the 4 for the lower value
- two suits of the 4 for the upper value
- a last card which is any of the $52 - 8$ cards which don't have a value of the pair.

Again, I can frame this in terms of a bijection with a list of information. A hand with 5 cards in it is in bijection with a list containing 4 pieces of information. For instance the hand $3\heartsuit$, $3\clubsuit$, $7\clubsuit$, $K\spadesuit$, $K\clubsuit$ is a hand with two pairs. It is in bijection with $(\{3, K\}, \{\heartsuit, \clubsuit\}, \{\spadesuit, \clubsuit\}, 7\clubsuit)$.

Now the number of possible lists are easy to count by the multiplication principle. There are $\binom{13}{2}$ choices for the values of the pairs. There are $\binom{4}{2}$ possible sets of two suits from the set $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$ and there are 44 remaining cards. Therefore the number of hands with two pairs is

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 = 123,552 \ .$$

**pair:**
I note that every pair hand is determined by the following 4 pieces of information.

- the value of the card that appears twice
- the values of the other three cards (all different and not the same as the last value)
- the two suits used by the pair

- a suit used by the smallest of the three cards
- a suit used by the middle of the three cards
- a suit used by the largest of the three cards

That is I am saying that if I am given a particular five card hand with containing exactly a pair, then the 6 pieces of information are all that is necessary to determine the hand and the hand determines the information. Therefore the set of hands containing a pair are in bijection with tuples containing the information in that list. For example the hand $3\heartsuit$, $5\diamondsuit$, $7\diamondsuit$, $7\clubsuit$, $10\spadesuit$ and this is isomorphic to this list $(7, \{3, 5, 10\}, \{\diamondsuit, \clubsuit\}, \heartsuit, \diamondsuit, \spadesuit)$.

Now there are 13 ways of choosing the card that appears twice; $\binom{12}{3}$ ways of choosing a set of three elements from the 12 values that are not the pair; there are $\binom{4}{2}$ possible sets for the suits which appear in the pair; there are 4 suits possible for the non-pair card; 4 suits for the second non-pair card; 4 suits for the third non-pair card. In total there are

$$13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4 \cdot 4 \cdot 4 = 1,098,240$$

is the number of hands with exactly one pair.

**high card:**
A high card hand has 5 different values that do not form a straight and 5 suits which do not form a flush. Therefore there are $\left(\binom{13}{5} - 10\right)\left(4^5 - 4\right) = 1,302,540$.

If we did this all right then the sum of all of the categories above is equal to $\binom{52}{5}$ which is the number of 5 card hands in total. Lets do the check

| | |
|---|---|
| straight flush | 40 |
| 4 of a kind | 624 |
| full house | 3,744 |
| flush | 5,108 |
| straight | 10,200 |
| 3 of a kind | 54,912 |
| two pairs | 123,552 |
| pair | 1,098,240 |
| high card | 1,302,540 |
| total | 2,598,960 |

You can check on your calculator that $\binom{52}{5} = 2,598,960$. That is a *very* strong indication that every one of our explanations above is correct.