FIFTH LECTURE : SEPTEMBER 23, 2014

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I put up four identities that I discussed in previous lectures

$$x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k) (x)^{(k)}$$

$$x^n = \sum_{k=1}^{n} S(n,k) (x)_k$$

$$(x)^{(n)} = \sum_{k=1}^{n} s'(n,k) x^k$$

$$(x)_{(n)} = \sum_{k=1}^{n} (-1)^{n-k} s'(n,k) x^k$$

We will use only the first one for the moment. We will also use one other identity

$$(1)^{(k)} + (2)^{(k)} + \cdots + (n)^{(k)} = \sum_{i=1}^{n} (i)^{(k)} = \frac{(n)^{(k+1)}}{k+1}.$$

The goal is to give a formula for

$$1^r + 2^r + \cdots + n^r = \sum_{i=1}^{n} i^r = ???$$

Here goes:

$$1^r + 2^r + \cdots + n^r = \sum_{i=1}^{n} i^r \tag{1}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{r} (-1)^{r-k} S(r,k) (i)^{(k)} \tag{2}$$

$$= \sum_{k=1}^{r} (-1)^{r-k} S(r,k) \sum_{i=1}^{n} (i)^{(k)} \tag{3}$$

$$= \sum_{k=1}^{r} (-1)^{r-k} S(r,k) \frac{(n)^{(k+1)}}{k+1} \tag{4}$$

Done! Lets see how this works. The values of $S(n,k) = S(n-1,k-1) + kS(n-1,k)$. Recall that we have the following table of values of $S(n,k)$.

$$\begin{array}{llllll}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
1 & 15 & 25 & 10 & 1 \\
1 & 31 & 90 & 65 & 15 & 1 \\
\vdots
\end{array}$$

$$\begin{aligned}
1+2+\cdots+n &= \frac{(n)^{(2)}}{2} = \frac{n(n+1)}{2} \\
1^2+2^2+\cdots+n^2 &= -\frac{(n)^{(2)}}{2} + \frac{(n)^{(3)}}{3} = \frac{n(n+1)(2n+1)}{6} \\
1^3+2^3+\cdots+n^3 &= \frac{(n)^{(2)}}{2} - 3\frac{(n)^{(3)}}{3} + \frac{(n)^{(4)}}{4} = \frac{n^2(n+1)^2}{4} \\
1^4+2^4+\cdots+n^4 &= -\frac{(n)^{(2)}}{2} + 7\frac{(n)^{(3)}}{3} - 6\frac{(n)^{(4)}}{4} + \frac{(n)^{(5)}}{5} \\
1^5+2^5+\cdots+n^5 &= \frac{(n)^{(2)}}{2} - 15\frac{(n)^{(3)}}{3} + 25\frac{(n)^{(4)}}{4} - 10\frac{(n)^{(5)}}{5} + \frac{(n)^{(6)}}{6}
\end{aligned}$$

We could simplify the last two, but if you just want a value, it is much easier to compute the rising factorial of n. I used a computer to do this and I found that

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

It may seem that we have just replaced on sum with another (which we have), but the advantage of the right hand side is that the number of terms in the sum only depends on the exponent and not on the value of n (which could be very large).

I cleared up the problem that I had with the explanation of the rising factorial. I explained why $n(n+1)(n+2) = n^3+3n^2+2n$ using combinatorics rather than algebra. It seems like a stupid thing to do because the algebra is much, much simpler, however there will be some identities where the explanation (all using words) is more powerful than the algebra techniques that we have. I had already corrected that in the notes for lecture on September 18 so I don't include it here.

I talked about the problem of counting the number of ways of distributing k objects to n people. It turns out that the answer depends on if the objects are distinct or not.

I first explained this by example. If I was distributing a nickel, a dime and a quarter to 2 different people, then I wrote down the 8 possibilities as:

$$(NDQ|-), (DQ|N), (NQ|D), (ND|Q), (N|DQ), (D|NQ), (Q|ND), (-|NDQ)\ .$$

I didn't bother to write down the answer in the case of 3 different people but it was easy to see what was happening in general, if there are n different people then there are n^k ways of distributing the k distinct objects because each of the k objects have n choices in how they are distributed.

But if the objects are not distinct, say that I have three quarters then I could distribute them in only 4 different ways:

$$(3Q|0Q), (2Q|1Q), (1Q|2Q), (0Q|3Q)\ .$$

So there seems to be a different formula. If there were three different people then the number of ways of distributing 3 quarters is 10:

$$(3Q|0Q|0Q), (2Q|1Q|0Q), (1Q|2Q|0Q), (0Q|3Q|0Q), (2Q|0Q|1Q),$$

$$(1Q|1Q|1Q), (0Q|2Q|1Q), (1Q|0Q|2Q), (0Q|1Q|2Q), (0Q|0Q|3Q)\ .$$

What is the formula? I wrote down a table by computing examples and then asked you to guess at the answer.

n/k	1	2	3	4	5
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	
3	1	4	10		
4	1	5			

We may not have drawn the table for $\binom{n}{k}$, but you have probably seen it before. It is sometimes called Pascal's triangle.

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

The number of ways that k objects can be distributed to n people is equal to $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

The explanation is that there is a bijection between distributions of k objects to n people and sequences with k dots and $n-1$ bars.

$$(3Q|0Q|0Q) \leftrightarrow \bullet\bullet\bullet||$$

$$(2Q|1Q|0Q) \leftrightarrow \bullet\bullet|\bullet|$$
$$(1Q|2Q|0Q) \leftrightarrow \bullet|\bullet\bullet|$$
$$(0Q|3Q|0Q) \leftrightarrow |\bullet\bullet\bullet|$$
$$(2Q|0Q|1Q) \leftrightarrow \bullet\bullet||\bullet$$
$$(1Q|1Q|1Q) \leftrightarrow \bullet|\bullet|\bullet$$
$$(0Q|2Q|1Q) \leftrightarrow |\bullet\bullet|\bullet$$
$$(1Q|0Q|2Q) \leftrightarrow \bullet||\bullet\bullet$$
$$(0Q|1Q|2Q) \leftrightarrow |\bullet|\bullet\bullet$$
$$(0Q|0Q|3Q) \leftrightarrow ||\bullet\bullet\bullet$$

Notice that we get every sequence of length 5 with $n-1$ bars (the separators between the people) and k dots (the k objects).

Now we know that there are $\binom{n+k-1}{k}$ because we can choose the k positions of the dots from the $(n-1)+k$ total spots (and the rest are filled with bars). This is also equal to $\binom{n+k-1}{n-1}$ because we can choose the $n-1$ positions of the bars and fill the rest with dots.

I then finished by stating the following problem:

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}$$

In the case of $n=3$ it says that

$$\binom{6}{3}^2 = 400 = \binom{6}{0}\binom{6}{0}\binom{6}{3} + \binom{6}{1}\binom{5}{1}\binom{4}{2} + \binom{6}{2}\binom{4}{2}\binom{2}{1} + \binom{6}{3}\binom{3}{3}\binom{0}{0}$$

Note that the identity that I put on the board in class might have had an error in it, but this one is correct. Next time I will give and explanation of why this is true in words (and it is exactly like one of the problems that I give you on your homework).