HOMEWORK #1 - MATH 4160

ASSIGNED: THURSDAY, SEPTEMBER 21, 2017
DUE: THURSDAY, OCTOBER 5, 2017

Write your homework solutions neatly and clearly and in LaTeX. Provide full explanations and justify all of your answers.

(1) Prove: For $k \geq 1$ and $n \geq 0$,

$$\sum_{i=1}^{n} (i)_k = (1)_k + (2)_k + \cdots + (n)_k = \frac{(n+1)_{k+1}}{k+1} \ .$$

(2) Prove: for $n \geq 0$,

$$1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4} \ .$$

(3) Prove: for $n \geq 0$,

$$1^4 + 2^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \ .$$

(4) The (unsigned) Stirling numbers of the first kind are defined as $s'(n,1) = (n-1)!$ for $n \geq 1$, $s'(n,n) = 1$ for $n \geq 0$, $s'(n,k) = 0$ if $k > n$ or $k \leq 0$ and for $2 \leq k \leq n-1$,

$$s'(n,k) = (n-1)s'(n-1,k) + s'(n-1,k-1) \ .$$

In class, we defined for $k > 0$, $(x)_k := x(x-1)(x-2)\cdots(x-k+1)$ (falling factorial) and there is also the $(x)^{(k)} := x(x+1)(x+2)\cdots(x+k-1)$ (rising factorial). Show that

$$(x)^{(n)} = \sum_{k=1}^{n} s'(n,k)x^k \ .$$

(5) The Stirling numbers of the second kind are defined as $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$ for $n \geq 0$, $\left\{ {0 \atop 0} \right\} = 1$, $\left\{ {n \atop k} \right\} = 0$ if $k > n$ or $k \leq 0$, and for $2 \leq k \leq n-1$,

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} \ .$$

Prove that

$$x^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} (x)_k \ .$$

(6) For $n \geq 0$ and $1 \leq k \leq n$, let $S(n,k) =$ the number set partitions of $\{1, 2, \ldots, n\}$ that have k disjoint sets.

(a) Explain why $S(n,1) = 1$ and why $S(n,n) = 1$.

(b) In 2-3 sentences, explain why for $2 \leq k \leq n$,

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)\ .$$

(c) Use parts (a) and (b) to show that $S(n,k) = \left\{ {n \atop k} \right\}$ for all $n \geq 1$ and all $1 \leq k \leq n$.

Enumeration problems. The emphasis is on you coming up with a clear explanation of why the answer is true. I more care why the answer is than what the answer is.

(1) An urn contains nine red balls, eight green balls and six blue balls. Five balls are randomly chosen from the urn. Find the probability that the result of that choice will have at least one of each color.
(2) How many three of a kind poker hands are there from a deck of 50 cards because it has the five and nine of spades missing? A is high or low.
(3) How many permutations of 8 have exactly one descent? (for a permutation $\pi = a_1 a_2 \ldots a_8$, a descent is a position i with $1 \leq i < 8$ such that $a_i > a_{i+1}$)

Combinatorial proofs: For the following problems give a combinatorial proof by describing a set that is counted by the left hand side of the equality and a set that is counted by the right hand side of the equality and explaining why these two sets are the same.

(1)

$$\binom{2n+1}{2} = 2\binom{n+1}{2} + n^2$$

(2)

$$\binom{2n}{n}\binom{2n}{n+1} = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{n-k}\binom{n}{n+1-k}$$