HOMEWORK #3 - MATH 4160

ASSIGNED: OCT 31, 2017 DUE: NOV 13, 2017

Write your homework solutions neatly and clearly (type!). Provide full explanations and justify all of your answers.

(1) Give a proof of the following identity by counting two sets of partitions in two different ways.

$$\prod_{i\geq 0}(1+x^{2i+1}) = 1 + \sum_{n\geq 1} x^{n^2} \prod_{j=1}^{n} \frac{1}{1-x^{2j}}$$

(2) Find the exponential generating function for the number of words of length n using letters a, b, c, d, e such that
   (a) all 5 letters occur without restriction
   (b) at least three of the letters a, b, c appears
   (c) the letters a, b, c each appear at least twice and c, d, e each appear an even number of times
   (d) the letters a, b, c each appear at least twice and c, d, e together appear an even number of times

   Use your generating function to find the number of number of words of length 20 with the restrictions above.
   As a hint, for length 10 the number of words for part (a) is 9765625, (b) 9645561, (c) 337470, (d) 953190,

(3) In class we found the exponential generating function for the Bell numbers B_n which are defined by the recurrence $B_0 = 1$, $B_1 = 1$ and $B_{n+1} = \sum_{i=1}^{n} \binom{n}{i} B_{n-i}$ for all $n \geq 1$. We found that $B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{e^x-1}$. Recall that the Stirling numbers of the second kind $S_{n,k}$ are defined as the number of set partitions into k parts. They are defined recursively as $S_{0,0} = 1$, $S_{n,1} = S_{n,n} = 1$ for all $n \geq 1$, and $S_{n,k} = 0$ if $k > n$. Moreover $S_{n+1,k} = kS_{n,k} + S_{n,k-1}$ for $n \geq 0$ and $1 \leq k \leq n$.

   Refine the computation that gives the formula for $B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{e^x-1}$ (that is we show that $B(x)$ satisfies a differential equation and $B(0) = B_0$ and $B'(0) = B_1$) to show that

$$\mathbb{S}(x,q) = \sum_{n\geq 0}\sum_{k\geq 0} S_{n,k} q^k \frac{x^n}{n!} = e^{q(e^x-1)}.$$

(4) Use the result of the previous problem to give the generating function for the number of set partitions into an odd number of parts. That is, if we let

$$B_n^o = \sum_{k=0}^{\lceil n/2\rceil-1} S_{n,2k+1},$$

   then find a formula for $B^o(x) = \sum_{n\geq 0} B_n^o \frac{x^n}{n!}$.

(5) Given the generating function, $A(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$, find a formula for the generating function

$$\tilde{A}(x) = a_1 + a_0 x + a_3 x^2 + a_2 x^3 + a_5 x^4 + a_4 x^5 + \cdots$$

in terms of $A(x)$.