EXPONENTIAL GENERATING FUNCTION EXAMPLE

MIKE ZABROCKI - OCTOBER 19, 2017

I wanted to show an example of what I expected for a problem on generating functions for the homework. I showed an example of an exponential generating function instead of an ordinary generating function because the principle was the same and you had something to compare how exponential and ordinary generating functions differ.

Let me state precisely the multiplication principle of exponential generating functions.

Theorem 1. *(the multiplication principle of exponential generating functions) If* $A(x) = \sum_{n \geq 0} a_n \frac{x_n}{n!}$ *and* $B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$ *then the coefficient of* $\frac{x^n}{n!}$ *in* $A(x)B(x)$ *is equal to*

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \ .$$

In combinatorial terms if a_n *is equal to the number of widgets of size* n *and* b_n *is equal to the number of doodles of size* n*, then* $A(x)B(x)$ *is equal to the exponential generating function for the number of triples* (S, x, y) *where* x *is a widget of size* k*,* y *is a doodle of size* $n-k$ *and* S *is a subset of* $\{1, 2, \ldots, n\}$ *of size* k*.*

This is stated more generally in the notes that I wrote in 2012.

Theorem 2. *(The Multiplication Principle of Exponential Generating Functions) Let* $A_i(x) = \sum_{n \geq 0} a_n^{(i)} \frac{x^n}{n!}$*, then*

$$A_1(x)A_2(x)\cdots A_d(x) = \sum_{n \geq 0} \left(\sum_{i_1+i_2+\cdots+i_d=n} \binom{n}{i_1, i_2, \ldots, i_d} a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_d}^{(d)} \right) \frac{x^n}{n!} \ .$$

Alternatively the coefficient of $\frac{x^n}{n!}$ *in* $A_1(x)A_2(x)\cdots A_d(x)$ *is equal to*

$$\sum_{i_1+i_2+\cdots+i_d=n} \binom{n}{i_1, i_2, \ldots, i_d} a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_d}^{(d)} \ .$$

I took a problem that was extremely similar to one of the homework problems, but differed because it was enumerating a set of words (structures with order) rather than distributions of loonies and twonies to people. Please refer to Homework # 2 problem 5 for the comparison.

Count the number of words of length 20 with the letters A, B, C, D satisfying the following properties.

(1) no restriction
(2) there are an even number of A's and B's in total
(3) there are at most a total of 6 A's and Bs.
(4) the number of A's and B's is even and there are at most 6.
(5) the number of A's and B's is even or there are at most 6 in total.

Lemma 3. *The exponential generating function for the number of words of length n with C's and D's (or A's and B's) is equal to $e^{2x} = \sum_{n\geq 0} 2^n \frac{x^n}{n!}$.*

Proof. There is one word of length n with C's only (or D's only). The exponential generating function for the number of words is equal to $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$. Every word of length n with C's and D's is isomorphic to the number of triples (S, x, y) consisting of a subset S of $\{1, 2, \ldots, n\}$ representing the positions of the C's, a word x of length $|S|$ of C's and a word y of D's of length $n-k$. Therefore the exponential generating function for the number of these words will be the product of exponential generating functions for the words of C's and the exponential generating functions for the words of D's. Their product is $e^x e^x = e^{2x}$. □

or better

Proof. There are 2^n words of length n with C's and D's because for each letter of the word there are two choices. Therefore the exponential generating function is $\sum_{n\geq 0} 2^n \frac{x^n}{n!} = e^{2x}$. □

Lemma 4. *The exponential generating function for the number of words with A's and Bs where there an even number of A's and B's is $(e^{2x} + e^{-2x})/2$.*

Proof. The exponential generating function for the words with A's and B's with an even number of letters is equal to the terms with even exponents in the exponential generating function e^{2x} for all words with A's and B's. This is $(e^{2x} + e^{-2x})/2$. □

Lemma 5. *The exponential generating function for the number of words with A's and Bs with at most 6 letters in total is $1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6$.*

Proof. A Sage calculation yields the first 6 terms of the exponential generating function for the words with A's and B's, namely $e^{2x} = 1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6 + \cdots$. □

Lemma 6. *The exponential generating function for the number of words with A's and Bs with at most 6 letters in total and an even number is $1 + 2x^2 + 2/3x^4 + 4/45x^6$.*

Proof. The even terms from the expression in Lemma 5 is the exponential generating function for the number of terms of even length and length less than or equal to 6. □

Lemma 7. *The exponential generating function for the number of words with A's and Bs with at most 6 letters in total or an even number is $(e^{2x} + e^{-2x})/2 + 2x + 4/3x^3 + 4/15x^5$.*

Proof. The odd terms from the expression in Lemma 5 is equal to the exponential generating function for the words which are of length less than or equal to 6 and of odd length and this is equal to $2x + 4/3x^3 + 4/15x^5$. The disjoint union of this set of words and those that are of even length is equal to the words with A's and Bs with at most 6 letters in total or an even number. By the addition principle of exponential generating functions, this is equal to the sum of the expression for the exponential generating function for the number of words with A's and Bs where there an even number of A's and B's and the exponential generating function for the words which are of length less than or equal to 6 and of odd length and this is equal to $(e^{2x} + e^{-2x})/2 + 2x + 4/3x^3 + 4/15x^5$. □

Note that now a lot of my answer is cut and paste:

Answer to part (1): Therefore the exponential generating function for the number of words with the letters A, B, C, and D is equal to the product of the exponential generating functions for the words with A's and B's times the exponential generating functions for the words with C's and D's. Their product is $e^{2x}e^{2x} = e^{4x}$.

The number of words with the letters A, B, C, and D of length 20 is equal to 20! times the coefficient of x^{20} in e^{4x} which is equal to 4^{20}.

Answer to part (2): The exponential generating function for the number of words with the letters A, B, C, D where there are an even number of A's and B's is equal to the product of the exponential generating function for the number of words with A's and Bs where there an even number of A's and B's (from Lemma 4) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}(e^{2x} - e^{-2x})/2 = (e^{4x} - 1)/2$.

The number of words with the letters A, B, C, and D where there are an even number of A's and B's of length 20 is equal to 20! times the coefficient of x^{20} in $(e^{4x} - 1)/2$ which is equal to $4^{20}/2$.

Answer to part (3): The exponential generating function for the number of words with the letters A, B, C, D where there are at most 6 A's and B's in total is equal to the product of the exponential generating function for the number of words with A's and Bs where there at most 6 A's and B's (from Lemma 5) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}(1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6)$.

The number of words with the letters A, B, C, and D where there are at most 6 A's and B's of length 20 is equal to 20! times the coefficient of x^{20} in $e^{2x}(1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6)$ which (according to Sage) is equal to 63396904960.

Answer to part (4): The exponential generating function for the number of words with the letters A, B, C, D where there are at most 6 A's and B's in total and an even number is equal to the product of the exponential generating function for the number of words with A's and Bs where there at most 6 A's and B's and an even number (from Lemma 6) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}(1 + 2x^2 + 2/3x^4 + 4/45x^6)$.

The number of words with the letters A, B, C, and D where there are at most 6 A's and B's and an even number of length 20 is equal to 20! times the coefficient of x^{20} in $e^{2x}(1 + 2x^2 + 2/3x^4 + 4/45x^6)$ which (according to Sage) is equal to 45923434496.

Answer to part (5): The exponential generating function for the number of words with the letters A, B, C, D where there are less than or equal to 6 A's and B's in total or an even number of A's and B's is equal to the product of the exponential generating function for the number of words with A's and Bs where there at most 6 A's and B's or an even number (from Lemma 7) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}((e^{2x}+e^{-2x})/2+2x+4/3x^3+4/15x^5)$.

The number of words with the letters A, B, C, and D where there are at less than or equal to 6 A's and B's in total or an even number of A's and B's of length 20 is equal to 20! times the coefficient of x^{20} in $e^{2x}((e^{2x}+e^{-2x})/2+2x+4/3x^3+4/15x^5)$ which (according to Sage) is equal to 567229284352.