NOTES FROM THE FIRST CLASS

MIKE ZABROCKI - SEPTEMBER 9, 2014

The course web page and description is at

`http://garsia.math.yorku.ca/ zabrocki/math4160f14/`

The first thing I did was try to explain what combinatorics is about and what we will learn in this class. The main takeaway message is "combinatorics = counting" and it has applications through all types of mathematics. My research is in the area of algebraic combinatorics where I use the techniques of studying discrete structures in algebraic constructions such as modules, algebras, groups and rings. These sorts of questions lead to beautiful mathematics.

I mentioned that one of the things that is great about combinatorics as a subject is that it is "elementary." That is, that it is based on very little background knowledge of other mathematics. The word elementary in mathematics does *not* mean that the subject is easy, just that it is possible to understand the mathematics from very few principles.

The course will consist of homework assignments and take home exams. I would encourage you to work together on the homework assignments, but I want you to hand in your own work (and not just copying). FYI, the last time I taught this class two students went to see the associate dean over concerns of cheating. Once I feel that there is an issue it will be for their office to resolve.

The take home exams will be similar to the homework. On these I will be explicit that I expect you to work alone. You can come and ask me questions and I will help as best as I can, but I don't want you to discuss with your classmates or online resources.

I started then talking about combinatorics and about three tools that we will use.

1. *The equality principle* - If there exists a bijection between two sets A and B then $|A| = |B|$ (note that $|A|$ is the symbol I will use for the number of elements in the set A).

Example: Consider "the set of subsets of $\{1, 2, \ldots, n\}$ that contain both 1 and n" and "the set of subsets of $\{1, 2, \ldots, n-2\}$. I claim that there is a bijection between both of these two sets because if we take $\{1, n, a_1, a_2, \ldots, a_k\}$ as a set which contains both 1 and n and the $2 \leq a_i \leq n-1$ then this is sent to set $\{a_1 - 1, a_2 - 1, \ldots, a_k - 1\}$ is a subset of $\{1, 2, \ldots, n-2\}$.

In the case of $n = 4$ we see that there are four subsets which contain $\{1, 4\}$, namely $\{\{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}$. There are also four subsets of $\{1, 2\}$, namely $\{\{\}$, $\{1\}$, $\{2\}$, $\{1, 2\}\}$.

I will often not state that I am using this principle, just that one description is equal to another and then so "clearly" these two sets have the same number of elements are equal. The word *clearly* is loaded. What it means is that there is some detail to be understood at this point and you need to figure it out (and hence it probably isn't clear in any way).

2. *The addition principle* - If there are three sets related by $A = B \uplus C$ (which means A is the union of B and C and both B and C don't have elements in common), then $|A| = |B| + |C|$.

Example: Lets set $\binom{n}{k}$ to be a symbol which represents the number of subsets of an n element set with exactly k elements. So, for instance $\binom{4}{2} = |\{\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\}\}|$ Consider $n \geq k \geq 1$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}\ .$$

Proof: Every k element subset of the set $\{1,2,\ldots,n\}$ either contains n or it does not. If the set contains n, then it has $k-1$ other elements from $\{1,2,\ldots,n-1\}$. Otherwise, the set is has k elements from $\{1,2,\ldots,n-1\}$.

3. *The multiplication principle* - If the set A consists of all pairs (x,y) where x is an element of B and y is an element of C, then $|A| = |B| \cdot |C|$.

Example: Again consider the case when $n \geq k \geq 1$, then

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$$

Proof: The left hand side of this equation represents the number of pairs consisting of a subset of $\{1,2,\ldots,n\}$ of with k elements and a choice of one of the k elements which will be colored orange.

The right hand side consists of all pairs whose first element is one of the numbers a where $1 \leq a \leq n$ to painted orange, followed by a $k-1$ element subset of $\{1,2,\ldots,n\}\backslash\{a\}$.

NOTES FROM THE SECOND CLASS

MIKE ZABROCKI - SEPTEMBER 11, 2014

In the first class we discussed three tools. Let me restate them again here (in a more general form).

(1) the equality principle:
    If there is a bijection between a finite set A and a finite set B, then they have the same number of elements.
(2) the addition principle:
    Say there are sets $A_1, A_2, \ldots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \cup A_2 \cup \cdots \cup A_n$ is
$$a_1 + a_2 + a_3 + \cdots + a_n$$
(3) multiplication principle
    say there are sets $A_1, A_2, \ldots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n)$ where $x_i \in A_i\}$ is $a_1 a_2 \cdots a_n$

Application:
$S(n, k) =$ the number of set partitions of $\{1, 2, \ldots, n\}$ into k subsets
E.g.
$$\{123\}$$
$$\{12, 3\}, \{13, 2\}, \{1, 23\}$$
$$\{1, 2, 3\}$$

$$\{1234\}$$
$$\{123, 4\}, \{124, 3\}, \{134, 2\}, \{234, 1\}, \{12, 34\}, \{13, 24\}, \{14, 23\}$$
$$\{12, 3, 4\}, \{13, 2, 4\}, \{14, 2, 3\}, \{23, 1, 4\}, \{24, 1, 3\}, \{34, 1, 2\}$$
$$\{1, 2, 3, 4\}$$

1			
1	1		
1	3	1	
1	7	6	1

but I can't do more of this table by hand because it there are too many set partitions of 5.

So let me argue the following using the three principles we start this class with.

All set partitions of $\{1, 2, \ldots, n\}$ into k parts = the set partitions where n is by itself into $k-1$ other parts union the set partitions where n is with one of the other k parts of $\{1, 2, , n-1\}$ so

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)\ .$$

This allows us to compute the table of values of $S(n,k)$ much further than we did before without actually counting each individual one.

$$\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
1 & 15 & 25 & 10 & 1 \\
1 & 31 & 90 & 65 & 15 & 1 \\
\vdots
\end{array}$$

What we would like to do is start with

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$

and then to generalize this and get to

$$1^r + 2^r + \cdots + n^r = ???$$

Just to show what we are up against:

$$1 + 2 + 3 + \cdots + n = n(n+1)/2 \tag{1}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6 \tag{2}$$

$$1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4 \tag{3}$$

$$1^4 + 2^4 + \cdots + n^4 = ??? \tag{4}$$

I don't even know what the right hand side is for the last of these equations.

I showed a technique for demonstrating equalities like the one above, but this technique only works if you know the right hand side. I showed the following general trick called 'telescoping sums.'

In order to show that

$$a(1) + a(2) + \cdots + a(n) = b(n)$$

for some formulas $a(n)$ and $b(n)$ and $b(0) = 0$, then all you need to do is show that $b(n) - b(n-1) = a(n)$. If you do then

$$b(n) - b(n-1) = a(n)$$

$$b(n-1) - b(n-2) = a(n-1)$$

$$\vdots$$
$$b(2) - b(1) = a(2)$$
$$b(1) - b(0) = a(1)$$

Now add up all the terms on the left hand side and we have

$$(b(n) - b(n-1)) + (b(n-1) - b(n-2)) + \cdots + (b(1) - b(0)) = b(n) - b(0) = b(n)$$

If you add up all the terms on the right hand side of the equality then you have

$$a(n) + a(n-1) + \cdots + a(2) + a(1)$$

and they must be equal.

But there is a sequence of equations that continues (unlike equations (1)-(4)):

$$1 + 2 + 3 + \cdots + n = n(n+1)/2(\textbf{??}) \tag{5}$$

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = (n+1)n(n-1)/3 \tag{6}$$

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)/4 \tag{7}$$

$$\vdots$$

$$1 \cdot 2 \cdots k + 2 \cdot 3 \cdots (k+1) + \cdots + n \cdot (n+1) \cdots (n+k-1) = n \cdot (n+1) \cdots (n+k)/(k+1) \tag{8}$$

You should be able to prove this entire sequence of equations either by (a) induction (on n) or (b) telescoping sums.

By telescoping sums, you need only do the computation,

$$\frac{1}{k+1} n(n+1)(n+2)\cdots(n+k) - \frac{1}{k+1}(n-1)n(n+1)\cdots(n+k-1) =$$
$$\frac{1}{k+1}((n+k) - (n-1))n(n+1)\cdots(n+k-2) = n(n+1)\cdots(n+k-1)$$

Therefore, by the method of telescoping sums, (8) follows and all the equations (**??**)-(8) are special cases of this one.

Define for k and integer with $k > 0$, set:

$$(x)^{(k)} = x(x+1)(x+2)\cdots(x+k-1)$$

such that there are k terms in the product.

Examples: $(x)^{(1)} = x$, $(x)^{(2)} = x(x+1)$, $(x)^{(3)} = x(x+1)(x+2), \ldots$

This is new notation that makes some of our formulas simpler. Equations (**??**) - (8) are now

$$(1)^{(1)} + (2)^{(1)} + \cdots + (n)^{(1)} = \frac{(n)^{(2)}}{2} \tag{9}$$

$$(1)^{(1)} + (2)^{(2)} + \cdots + (n)^{(2)} = \frac{(n)^{(3)}}{3} \tag{10}$$

$$(1)^{(3)} + (2)^{(3)} + \cdots + (n)^{(3)} = \frac{(n)^{(4)}}{4} \tag{11}$$

$$\vdots$$

$$(1)^{(k)} + (2)^{(k)} + \cdots + (n)^{(k)} = \frac{(n)^{(k+1)}}{k+1} \tag{12}$$

Now it arises that the table of numbers $S(n,k)$ appear in the expansion of x^n in terms of $(x)_k$. In particular we have

$$x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)} \tag{13}$$

Example:

$$\begin{aligned}
(x)^{(1)} &= x^1 \\
-(x)^{(1)} + (x)^{(2)} &= -x + x(x+1) = -x + x^2 + x = x^2 \\
(x)^{(1)} - 3(x)^{(2)} + (x)^{(3)} &= x - 3(x^2+x) + (x^3+3x^2+2x) = x^3 \\
-(x)^{(1)} + 7(x)^{(2)} - 6(x)^{(3)} + (x)^{(4)} &= -x + 7x(x+1) - 6x(x+1)(x+2) + x(x+1)(x+2)(x+3) \\
&= -x + 7(x^2+x) - 6(x^3+3x^2+2x) + x^4 + 6x^3 + 11x^2 + 6x \\
&= x^4
\end{aligned}$$

So it should seem surprising that it is even possible to give a formula for x^n in terms of $(x)^{(k)}$, and hopefully it is even more surprising that these coefficients are counted by combinatorial objects called set partitions.

THIRD LECTURE : SEPTEMBER 18, 2014

MIKE ZABROCKI

I started off by listing the building block numbers that we have already seen and their combinatorial interpretations.

Theorem: For $n \geq 1$, $x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}$.

Proof: Note that

$$x^1 = S(1,1)(x)^{(1)}$$

Assume that

$$x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}$$

is true for some fixed n. Then if we compute x^{n+1} using this assumption, then

$$x^{n+1} = x^n \cdot x \tag{1}$$

$$= \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)} \cdot x \tag{2}$$

$$= \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}(x+k-k) \tag{3}$$

$$= \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}(x-k) - \sum_{k=1}^{n} kS(n,k)(x)^{(k)} \tag{4}$$

$$= \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k+1)} - \sum_{k=1}^{n} kS(n,k)(x)^{(k)} \tag{5}$$

At this point we need to be able to add like terms and so we need to shift the indices of the sum so that the $(x)^{(k+1)}$ looks like a $(x)^{(k)}$. To do this we replace k with $k-1$ everywhere in the first sum. The $k=1$ becomes $k-1=1$ or $k=2$. The $k=n$ becomes $k-1=n$ or

$k = n+1$.

$$= \sum_{k=2}^{n+1} (-1)^{n-k+1} S(n,k-1)(x)^{(k)} - \sum_{k=1}^{n} (-1)^{n-k} k S(n,k)(x)^{(k)} \tag{6}$$

$$= S(n,n)(x)^{(n+1)} + \sum_{k=2}^{n} (-1)^{n-k+1} S(n,k-1)(x)^{(k)} \tag{7}$$
$$- \sum_{k=2}^{n} (-1)^{n-k} k S(n,k)(x)^{(k)} + (-1)^n S(n,1)(x)_1$$

$$= S(n,n)(x)^{(n+1)} + \sum_{k=2}^{n} (-1)^{n-k+1} (S(n,k-1) + kS(n,k))(x)^{(k)} \tag{8}$$
$$+ (-1)^n S(n,1)(x)^{(1)}$$

$$= S(n+1,n+1)(x)^{(n+1)} + \sum_{k=2}^{n} (-1)^{n-k+1} S(n+1,k)(x)^{(k)} \tag{9}$$
$$+ (-1)^n S(n+1,1)(x)^{(1)} \ .$$

Some comments about this calculation:

In the last step we used that $S(n,n) = S(n+1,n+1) = S(n,1) = S(n+1,1) = 1$. (but they are the same sum). From step (7) to (8) we broke off the $k = n+1$ term of the first sum and the $k = 1$ term of the second sum. From step (8) to (9) we applied the equation

$$S(n+1,k) = S(n,k-1) + kS(n,k)$$

with $n \to n+1$. Finally we notice that equation (9) is equal to (by looking that the sum there has all the terms), $= \sum_{k=1}^{n+1} (-1)^{n+1-k} S(n+1,k)(x)^{(k)}$.

We conclude that by induction, $x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}$ is true for all $n \geq 1$.

Introduced $s'(n,k)$ and stated the other relationships between falling and rising factorials.

- $S(n,k) =$ the number of set partitions of $\{1,2,\ldots,n\}$ into k parts
- $B(n) =$ the number of set partitions of $\{1,2,\ldots,n\}$
- $s'(n,k)$ the number of permutations of $\{1,2,\ldots,n\}$ that have k-cyles
- $P(n) = n! =$ the number of permutations of $\{1,2,\ldots,n\}$
- $\binom{n}{k} =$ the number of subsets of $\{1,2,\ldots,n\}$ that contain k elements

FOURTH LECTURE

MIKE ZABROCKI - SEPTEMBER 18, 2014

I started off by listing the building block numbers that we have already seen and their combinatorial interpretations.

- $S(n,k)$ = the number of set partitions of $\{1,2,\ldots,n\}$ into k parts
- $B(n)$ = the number of set partitions of $\{1,2,\ldots,n\}$
- $s'(n,k)$ the number of permutations of $\{1,2,\ldots,n\}$ that have k-cyles
- $P(n) = n!$ = the number of permutations of $\{1,2,\ldots,n\}$
- $\binom{n}{k}$ = the number of subsets of $\{1,2,\ldots,n\}$ that contain k elements

Then we still need combinatorial explanations for n^k, $(n)_k$ and $(n)^{(k)}$. You can be super creative when you do this or extremely boring and express it in terms of sets and lists.

After a little discussion I mentioned that we can imagine that we have n colors of paints and k ordered objects then by the multiplication principle if we pick one of the n colors of paints for the first one, n for the second, n for the third, etc. then the total number of ways of coloring those k ordered objects is n^k.

You can and should be more creative than I was. Imagine that you have you have k people at dinner and they each order one of n desserts. Since each person has n choices for the dessert then there are n^k possible ways that the desserts can be ordered.

The standard answer is

n^k = the number of sequences of length k whose entries are $\{1,2,\ldots,n\}$

I then noted that since $(n)^{(k)} = n(n+1)(n+2)\cdots(n+k-1) = (n+k-1)_k$ that we just need to come up with a combinatorial interpretation for $(n)_k$. It turns out that this was a mistake. We should have come up with a combinatorial interpretation for $(n)^{(k)}$ separately.

Let me tell you what the standard interpretations are and then I expect you to give a 1-2 line explanation of why that is the case.

$(n)^{(k)}$ = the number of sequences of length k whose i^{th} entry is between 1 and $n+i-1$ for $1 \leq i \leq k$

$(n)_k$ = the number of sequences of length k whose entries are in $\{1,2,\ldots,n\}$ where each entry is different

Then I stupidly tried to use the explanation of

$$(2)^{(3)} = 2(2+1)(2+2)24 = 2 \cdot 2^1 + 3 \cdot 2^2 + 2^3$$

(this is a specific case of the more general formula $(n)^{(3)} = 2 \cdot n^1 + 3 \cdot n^2 + n^3$ that I want you to explain for homework). Where I went wrong is that I need to to use the combinatorial interpretation for $(2)^{(3)}$ and not the one for the falling factorial (they both can be done, but using the interpretation for $(4)_3$ is less clear and more tricky. Let me show it here.

The number $(2)^{(3)} = 24$ is the number of sequences whose first entry is between 1 and 2, whose second entry is between 1 and 3 and whose third entry is between 1 and 4. The 24 sequences are

$$111, 112, 113, 114, 121, 122, 123, 124, 131, 132, 133, 134$$

$$211, 212, 213, 214, 221, 222, 223, 224, 231, 232, 233, 234$$

Now if we rewrite this an express emphasize where the 2^1, 2^2 and 2^3 are coming from then we see $(2)^{(3)} = \mathbf{2}(\mathbf{2}+1)(\mathbf{2}+2) = 2 \cdot \mathbf{2} + 2\mathbf{2}^2 + \mathbf{2}^2 + \mathbf{2}^3$.

We will take a bold face **2** to mean that the entry in the sequence is either a 1 or a 2 and the other numbers which appear to mean that it is a 3 or a 4.

- $2 \cdot \mathbf{2}$ is equal to the number of sequences whose last two entries are either a 3 or a 4 (they are 133, 134, 233 and 234)
- $2\mathbf{2}^2$ is equal to the number of sequences whose first two entries are 1 or 2 and the last one is either 3 or 4 (they are $113, 114, 123, 124, 213, 214, 223, 224$)
- $\mathbf{2}^2$ is equal to the number of sequences whose first and third entries are 1 or 2 and the middle one is 3 (they are $131, 132, 231, 232$)
- $\mathbf{2}^3$ represents the number of sequences all of whose entries are 1 or 2 (they are $111, 112, 121, 211, 122, 212, 221, 222$)

Since it is always the case that the 3 or 4 appear in the last two positions, every sequence in the interpretation of $(2)^{(3)}$ falls into one of these 4 categories and hence they must also sum to 24 and as a by the addition principle, $(2)^{(3)} = 2 \cdot 2^1 + 3 \cdot 2^2 + 2^3$.

But since I messed up that explanation I skipped to counting hands of cards. Poker is a card game played with a 52 card deck with 13 values for the cards and 4 suits. Poker hands are ranked by how common a hand is.

The types of poker hands are:

- straight flush : a sequence of 5 cards in order all with the same suit (there is also a royal flush, $10, J, Q, K, A$ all the same suit, but these are also straight flushes so there is no real reason to separate them)
- 4 of a kind - 4 cards of the same value and one extra card
- full house - a pair and a three of a kind
- flush - five cards all one suit not a straight

- straight - five cards whose values are in a 5 card sequence and it is not the case they all have the same suit
- 3 of a kind - three cards of the same value and two extra cards with different values
- two pairs
- pair
- none of the above (often called 'high card')

A really good exercise is to figure out a way of counting the number of each of these sets using only addition (like I said, this is sometimes the more complicated way of coming up with the answer) and then add them all up and check that they add up to $\binom{52}{5}$ (the number of ways of picking 5 cards from a deck of 52).

The categories that we describe as a 'poker hand' are not the only descriptions possible, because I could make up a description and call that a poker hand. For example a 'sandwich two pair' is a 5 card hand that contains two pairs such that the two pairs have the same two suits and the 5th card has a value which is between the two pairs and has a suit which is the same as the pairs (e.g. $4\heartsuit, 4\clubsuit, 9\heartsuit, 9\clubsuit, 6\clubsuit$ is an example of a sandwich two pair, but $4\heartsuit, 4\clubsuit, 9\heartsuit, 9\clubsuit, J\clubsuit$ and $4\heartsuit, 4\clubsuit, 9\heartsuit, 9\spadesuit, 6\heartsuit$ are not).

Here is how you count all of the numbers of different types of hands.

straight flush:
There are also 40 straight flush hands because there are 4 possible suits and 10 possible straights (that begin with A through 10 as the lowest card, the A is either a high or low card but not both).

4-of-a-kind:
For instance, there are $13 \times 48 = 624$ possible 4-of-a-kind hands because we can choose which value appears 4 times in a 4-of-a-kind hand plus one extra card from the remaining 48 cards in the deck. Therefore a straight flush beats a 4-of-a-kind because there are more 4-of-a-kind hands than straight flush.

full house:
A full house consists of a three-of-a-kind and a pair. To specify one of these hands we must know the value of the three-of-a-kind, the three suits which appear in the three-of-a-kind, the value of the pair, and the two suits which appear in the pair. There are 13 values for the three-of-a-kind, $\binom{4}{3}$ ways of specifying the suits, 12 values for the pair and $\binom{4}{2}$ ways of specifying the suits. By the multiplication principle there are $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3,744$.

flush:
A flush hand has 5 values which are all different and one of the four suits. Now if you pick the 5 values from the 13 possible you will still include those sequences that are straight

flushes, so there are $\binom{13}{5} - 10$ possible sets of values and 4 suits. In total there are $\left(\binom{13}{5} - 10\right) \cdot 4 = 5,108$ possible flush hands

straight:
If a hand is a straight but not a straight flush then there are 10 possible straights and there are 4^5 ways of picking a suit for each of the cards of the straight, BUT we have to subtract off the number of ways that all suits are the same. By the multiplication principle we know that there are $10 \cdot (4^5 - 4) = 10,200$ possible straight hands which don't have a flush.

3-of-a-kind:
This hand is determined by which value is repeated three times, the three suits that appear and then two other cards from the remaining 48 (because we remove all the cards that are of the same value as the 3-of-a-kind) which do not form a pair. Since there are $12 \cdot \binom{4}{2}$ possible ways of making a pair from the remaining 48 cards, there are $\left(\binom{48}{2} - 12 \cdot \binom{4}{2}\right) = 1,056$ two cards which do not form a pair. Alternatively, we can pick two values from the remaining 12 and then a suit for each of those cards so there are also $\binom{12}{2} \cdot 4^2 = 1,056$ ways of picking the pair. In total there are $13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2 = 54,912$ three of a kind hands.

2-pair:
I also counted the number of hands with exactly two pairs. The following information completely determines a hand that has a two pair.

- two values (an upper and a lower) which will each appear twice in the hand
- two suits of the 4 for the lower value
- two suits of the 4 for the upper value
- a last card which is any of the $52 - 8$ cards which don't have a value of the pair.

Again, I can frame this in terms of a bijection with a list of information. A hand with 5 cards in it is in bijection with a list containing 4 pieces of information. For instance the hand $3\heartsuit$, $3\clubsuit$, $7\clubsuit$, $K\spadesuit$, $K\clubsuit$ is a hand with two pairs. It is in bijection with $(\{3, K\}, \{\heartsuit, \clubsuit\}, \{\spadesuit, \clubsuit\}, 7\clubsuit)$.

Now the number of possible lists are easy to count by the multiplication principle. There are $\binom{13}{2}$ choices for the values of the pairs. There are $\binom{4}{2}$ possible sets of two suits from the set $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$ and there are 44 remaining cards. Therefore the number of hands with two pairs is

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 = 123,552 \ .$$

pair:
I note that every pair hand is determined by the following 4 pieces of information.

- the value of the card that appears twice
- the values of the other three cards (all different and not the same as the last value)
- the two suits used by the pair

- a suit used by the smallest of the three cards
- a suit used by the middle of the three cards
- a suit used by the largest of the three cards

That is I am saying that if I am given a particular five card hand with containing exactly a pair, then the 6 pieces of information are all that is necessary to determine the hand and the hand determines the information. Therefore the set of hands containing a pair are in bijection with tuples containing the information in that list. For example the hand 3♡, 5♢, 7♢, 7♣, 10♠ and this is isomorphic to this list $(7, \{3, 5, 10\}, \{\diamondsuit, \clubsuit\}, \heartsuit, \diamondsuit, \spadesuit)$.

Now there are 13 ways of choosing the card that appears twice; $\binom{12}{3}$ ways of choosing a set of three elements from the 12 values that are not the pair; there are $\binom{4}{2}$ possible sets for the suits which appear in the pair; there are 4 suits possible for the non-pair card; 4 suits for the second non-pair card; 4 suits for the third non-pair card. In total there are

$$13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4 \cdot 4 \cdot 4 = 1,098,240$$

is the number of hands with exactly one pair.

high card:
A high card hand has 5 different values that do not form a straight and 5 suits which do not form a flush. Therefore there are $\left(\binom{13}{5} - 10\right)\left(4^5 - 4\right) = 1,302,540$.

If we did this all right then the sum of all of the categories above is equal to $\binom{52}{5}$ which is the number of 5 card hands in total. Lets do the check

straight flush	40
4 of a kind	624
full house	3,744
flush	5,108
straight	10,200
3 of a kind	54,912
two pairs	123,552
pair	1,098,240
high card	1,302,540
total	2,598,960

You can check on your calculator that $\binom{52}{5} = 2,598,960$. That is a *very* strong indication that every one of our explanations above is correct.

FIFTH LECTURE : SEPTEMBER 23, 2014

MIKE ZABROCKI

I put up four identities that I discussed in previous lectures

$$x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}$$

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k$$

$$(x)^{(n)} = \sum_{k=1}^{n} s'(n,k)x^k$$

$$(x)_{(n)} = \sum_{k=1}^{n} (-1)^{n-k} s'(n,k)x^k$$

We will use only the first one for the moment. We will also use one other identity

$$(1)^{(k)} + (2)^{(k)} + \cdots + (n)^{(k)} = \sum_{i=1}^{n} (i)^{(k)} = \frac{(n)^{(k+1)}}{k+1}.$$

The goal is to give a formula for

$$1^r + 2^r + \cdots + n^r = \sum_{i=1}^{n} i^r = ???$$

Here goes:

$$1^r + 2^r + \cdots + n^r = \sum_{i=1}^{n} i^r \tag{1}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{r} (-1)^{r-k} S(r,k)(i)^{(k)} \tag{2}$$

$$= \sum_{k=1}^{r} (-1)^{r-k} S(r,k) \sum_{i=1}^{n} (i)^{(k)} \tag{3}$$

$$= \sum_{k=1}^{r} (-1)^{r-k} S(r,k) \frac{(n)^{(k+1)}}{k+1} \tag{4}$$

Done! Lets see how this works. The values of $S(n,k) = S(n-1,k-1) + kS(n-1,k)$. Recall that we have the following table of values of $S(n,k)$.

1					
1	1				
1	3	1			
1	7	6	1		
1	15	25	10	1	
1	31	90	65	15	1
$\vdots$					

$$\begin{aligned}
1+2+\cdots+n &= \frac{(n)^{(2)}}{2} = \frac{n(n+1)}{2}\\
1^2+2^2+\cdots+n^2 &= -\frac{(n)^{(2)}}{2} + \frac{(n)^{(3)}}{3} = \frac{n(n+1)(2n+1)}{6}\\
1^3+2^3+\cdots+n^3 &= \frac{(n)^{(2)}}{2} - 3\frac{(n)^{(3)}}{3} + \frac{(n)^{(4)}}{4} = \frac{n^2(n+1)^2}{4}\\
1^4+2^4+\cdots+n^4 &= -\frac{(n)^{(2)}}{2} + 7\frac{(n)^{(3)}}{3} - 6\frac{(n)^{(4)}}{4} + \frac{(n)^{(5)}}{5}\\
1^5+2^5+\cdots+n^5 &= \frac{(n)^{(2)}}{2} - 15\frac{(n)^{(3)}}{3} + 25\frac{(n)^{(4)}}{4} - 10\frac{(n)^{(5)}}{5} + \frac{(n)^{(6)}}{6}
\end{aligned}$$

We could simplify the last two, but if you just want a value, it is much easier to compute the rising factorial of n. I used a computer to do this and I found that

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

It may seem that we have just replaced on sum with another (which we have), but the advantage of the right hand side is that the number of terms in the sum only depends on the exponent and not on the value of n (which could be very large).

I cleared up the problem that I had with the explanation of the rising factorial. I explained why $n(n+1)(n+2) = n^3+3n^2+2n$ using combinatorics rather than algebra. It seems like a stupid thing to do because the algebra is much, much simpler, however there will be some identities where the explanation (all using words) is more powerful than the algebra techniques that we have. I had already corrected that in the notes for lecture on September 18 so I don't include it here.

I talked about the problem of counting the number of ways of distributing k objects to n people. It turns out that the answer depends on if the objects are distinct or not.

I first explained this by example. If I was distributing a nickel, a dime and a quarter to 2 different people, then I wrote down the 8 possibilities as:

$$(NDQ|-), (DQ|N), (NQ|D), (ND|Q), (N|DQ), (D|NQ), (Q|ND), (-|NDQ)\ .$$

I didn't bother to write down the answer in the case of 3 different people but it was easy to see what was happening in general, if there are n different people then there are n^k ways of distributing the k distinct objects because each of the k objects have n choices in how they are distributed.

But if the objects are not distinct, say that I have three quarters then I could distribute them in only 4 different ways:

$$(3Q|0Q), (2Q|1Q), (1Q|2Q), (0Q|3Q)\ .$$

So there seems to be a different formula. If there were three different people then the number of ways of distributing 3 quarters is 10:

$$(3Q|0Q|0Q), (2Q|1Q|0Q), (1Q|2Q|0Q), (0Q|3Q|0Q), (2Q|0Q|1Q),$$

$$(1Q|1Q|1Q), (0Q|2Q|1Q), (1Q|0Q|2Q), (0Q|1Q|2Q), (0Q|0Q|3Q)\ .$$

What is the formula? I wrote down a table by computing examples and then asked you to guess at the answer.

n/k	1	2	3	4	5
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	
3	1	4	10		
4	1	5			

We may not have drawn the table for $\binom{n}{k}$, but you have probably seen it before. It is sometimes called Pascal's triangle.

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

The number of ways that k objects can be distributed to n people is equal to $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

The explanation is that there is a bijection between distributions of k objects to n people and sequences with k dots and $n-1$ bars.

$$(3Q|0Q|0Q) \leftrightarrow \bullet\bullet\bullet||$$

$$(2Q|1Q|0Q) \leftrightarrow \bullet\bullet|\bullet|$$
$$(1Q|2Q|0Q) \leftrightarrow \bullet|\bullet\bullet|$$
$$(0Q|3Q|0Q) \leftrightarrow |\bullet\bullet\bullet|$$
$$(2Q|0Q|1Q) \leftrightarrow \bullet\bullet||\bullet$$
$$(1Q|1Q|1Q) \leftrightarrow \bullet|\bullet|\bullet$$
$$(0Q|2Q|1Q) \leftrightarrow |\bullet\bullet|\bullet$$
$$(1Q|0Q|2Q) \leftrightarrow \bullet||\bullet\bullet$$
$$(0Q|1Q|2Q) \leftrightarrow |\bullet|\bullet\bullet$$
$$(0Q|0Q|3Q) \leftrightarrow ||\bullet\bullet\bullet$$

Notice that we get every sequence of length 5 with $n-1$ bars (the separators between the people) and k dots (the k objects).

Now we know that there are $\binom{n+k-1}{k}$ because we can choose the k positions of the dots from the $(n-1)+k$ total spots (and the rest are filled with bars). This is also equal to $\binom{n+k-1}{n-1}$ because we can choose the $n-1$ positions of the bars and fill the rest with dots.

I then finished by stating the following problem:

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}$$

In the case of $n=3$ it says that

$$\binom{6}{3}^2 = 400 = \binom{6}{0}\binom{6}{0}\binom{6}{3} + \binom{6}{1}\binom{5}{1}\binom{4}{2} + \binom{6}{2}\binom{4}{2}\binom{2}{1} + \binom{6}{3}\binom{3}{3}\binom{0}{0}$$

Note that the identity that I put on the board in class might have had an error in it, but this one is correct. Next time I will give and explanation of why this is true in words (and it is exactly like one of the problems that I give you on your homework).

SIXTH LECTURE : SEPTEMBER 25, 2014

MIKE ZABROCKI

The definition of $\binom{n}{k}$ is the number of ways of picking k elements from an n element set. If $k < 0$ or $k > n$ this symbol is just the number 0. Recall that $n!$ is the number of ways of ordering n elements.

Proposition 1. *For $n \geq 0$ and $0 \leq k \leq n$,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \ .$$

Proof. Pick k elements to go first. Order the k elements. Order the last $n-k$ elements. By the multiplication principle, number of ways of doing this is $\binom{n}{k} k!(n-k)!$. Since this is also the number of ways of ordering all n elements it is equal to $n!$. Therefore, $\binom{n}{k} k!(n-k)! = n!$. □

But then I said, well we can also show this same result 'by induction.'

Lemma 2.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. Every way of picking k elements from an n element set either contains the largest element or it doesn't. Since a choice of the n elements that contains the largest consists of $k-1$ other elements from an $n-1$ element set, there are $\binom{n-1}{k-1}$ ways of doing this. There are also $\binom{n-1}{k}$ ways of picking k elements that do not contain the largest. By the addition principle, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. □

Proposition 3. *For $n \geq 0$ and $0 \leq k \leq n$,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. For the base case, we note that there is exactly one way of choosing 0 elements from an n element set, hence $\binom{n}{0} = 1$ and there is only one way of choosing n elements from an n element set so $\binom{n}{n} = 1$. In particular, $\binom{0}{0} = \frac{0!}{0!0!} = 1$, $\binom{1}{0} = \frac{1!}{0!1!} = 1$ and $\binom{1}{1} = \frac{1!}{1!0!} = 1$ so the statement we are trying to prove holds for $n = 0$ and $n = 1$.

Now assume that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for some fixed n and all $0 \leq k \leq n$. Then we have that for $1 \leq k \leq n$,

$$\begin{aligned}
\binom{n+1}{k} &= \binom{n}{k-1} + \binom{n}{k} \text{ (by Lemma 2)} \\
&= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \text{ (by the inductive hypothesis)} \\
&= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right) \text{ (algebra)} \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n+1}{k(n-k+1)} \text{ (more algebra)} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \text{ (algebra wins!)}
\end{aligned}$$

The case of $k = 0$ and $k = n + 1$ hold because of what appears in the first paragraph. Therefore by the P.M.I., $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $0 \leq k \leq n$ for all $n \geq 0$. $\square$

I showed at least one place where a combinatorial explanation is an incredibly powerful tool.

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}$$

In the case of $n = 0, 1, 2$ it says that

$$\binom{0}{0}^2 = 1 = \binom{0}{0}\binom{0}{0}\binom{0}{0}$$

$$\binom{2}{1}^2 = 4 = \binom{2}{0}\binom{2}{0}\binom{2}{1} + \binom{2}{1}\binom{1}{1}\binom{0}{0} = 2 + 2$$

$$\binom{4}{2}^2 = 36 = \binom{4}{0}\binom{4}{0}\binom{4}{2} + \binom{4}{1}\binom{3}{1}\binom{2}{1} + \binom{4}{2}\binom{2}{2}\binom{0}{0} = 6 + 24 + 6$$

Proof. The left hand side of this equation, $\binom{2n}{n}^2$, represents the number of ways of taking an urn with $2n$ labeled balls and choosing n of them to color red, then putting all of the balls back into the urn and reaching in and pulling out n of them to color blue.

Some of the balls that go through this procedure will have red and blue paint will be purple. Say that there are k balls which are colored red, then there will be k balls which are colored blue, and $n - k$ balls which are colored purple. The number of ways that this can happen is the same as the number of ways of reaching in the urn and pulling out k

balls to color red (in $\binom{2n}{k}$ ways), then from the remaining $2n-k$ balls picking k more to be blue (in $\binom{2n-k}{k}$ ways) and then picking $n-k$ from the remaining $2n-2k$ (in $\binom{2n-2k}{n-k}$ ways). By the multiplication principle there are $\binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}$ ways of having k red, k blue and $n-k$ purple balls with the rest unpainted. Since k can be any value between 0 and n, by the addition principle we have that the total number of ways of picking the balls and coloring them this way is

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k} .$$

□

We considered rearranging letters of a word. I looked at the number of rearrangements of the word ANNOTATE. Consider rearrangements of the letters like TNTAAOEN or NEONATAT. I said that the following procedure will determine the word

- pick two positions from 8 for the letter A
- pick one position from the remaining 6 for the letter E
- pick two positions from the remaining 5 for the letter N
- pick one position from the remaining 3 for the letter O

the remaining two positions of the word will be filled with T's. That the set of rearrangements of the word ANNOTATE is in bijection with the sequences of subsets of $\{1, 2, \ldots, 8\}$ consisting of a subset of size 2, a subset of size 1, a subset of size 2 and a subset of size 1.

For example the word TNTAAOEN is sent under this bijection to $(\{4, 5\}, \{7\}, \{2, 8\}, \{6\})$. The number of such sequences is equal to

$$\binom{8}{2}\binom{6}{1}\binom{5}{2}\binom{3}{1} = \frac{8!}{2!6!}\frac{6!}{1!5!}\frac{5!}{2!3!}\frac{3!}{1!2!} = \frac{8!}{2!1!2!1!}$$

For this we define the notation we will call the multi-choose or multinomial coefficient. We will define $\binom{n}{k_1,k_2,\cdots,k_r}$ to be the number of ways of picking subsets of size $k_1, k_2 \ldots, k_r$ from an n element set For a sequence of integers $k_1, k_2, \ldots, k_r \geq 0$ such that $k_1 + k_2 + \cdots + k_r \leq n$, then

$$\binom{n}{k_1, k_2, \cdots, k_r} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{r-1}}{k_r}$$
$$= \frac{n!}{k_1!k_2!\cdots k_r!(n-k_1-k_2-\cdots-k_r)!} .$$

If $k_1 + k_2 + \cdots + k_r > n$ then $\binom{n}{k_1,k_2,\cdots,k_r} = 0$.

There is another place where this coefficient arises. I assume that everyone is familiar with the binomial theorem which gives an expansion of $(1+x)^n$ in terms of the binomial coefficients $\binom{n}{k}$. We have

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

for example, we have in particular

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 + 0x^5 + 0x^6 + 0x^7 + \ldots$$

The multinomial coefficient is a generalization of these coefficients. In fact, we have

$$(1 + x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r\leq n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}$$

With so many unknowns in this equation it is hard to appreciate this formula. But try an example. I can use the computer and see that $(1 + x + y)^4 =$

$$1+4\,x+4\,y+6\,x^2+12\,xy+6\,y^2+4\,x^3+12\,x^2y+12\,xy^2+4\,y^3+x^4+4\,x^3y+6\,x^2y^2+4\,xy^3+y^4$$

I can use this formula to see that $\binom{4}{1,2} = \frac{4!}{1!2!1!} = 12$ and I see that the coefficient of xy^2 in this expression is 12. If I want to answer a question like what is the coefficient of $x^7y^3z^9$ in the expression $(1+x+y+z)^{40}$ then I have a formula for this value, it is $\binom{40}{7,3,9} = \frac{40!}{7!3!9!21!}$ just as the binomial theorem tells me the coefficient of x^{19} in $(1 + x)^{40}$ is $\binom{40}{19} = \frac{40!}{19!21!}$.

Remark 1: How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + \cdots + x_r = n?$$

Answer: $\binom{n+r-1}{n} = \binom{n+r-1}{r-1}$. Why? Think of a dots and bars argument and find a bijection from a solution to this equation represented as a sequence $(x_1, x_2, x_3, \ldots, x_r)$ and a sequence of n dots and $r - 1$ bars.

Remark 2: How many paths are there in a lattice grid from $(0, 0)$ to (n, m) with n steps $E = (1, 0)$ and m steps $N = (0, 1)$?
Answer: $\binom{n+m}{n} = \binom{n+m}{m}$. Why? Think of a lattice path in a grid with N and E steps and translate it into a word of letters N and E such that there are m letters N and n letters E. The number of such words is determined by the number of ways of choosing the positions of the E steps in the word.

TWENTY-THRID LECTURE : NOVEMBER 27, 2014

MIKE ZABROCKI

I started with the problem "How many ways are there of putting 8 beads on a necklace using k colors assuming that adjacent beads cannot be the same color."

I found out that it is more difficult to count the colorings of necklaces with 8 beads with adjacent beads different colors than I thought and there was one step that I had wrong in my calculation that I did before class. I will put off the solution to this problem next time because I wanted to find a clever way to explain how to count these (the long way would have taken a long time).

The definition of a *partially ordered set* is a set S with a relation $\leq$ which has the properties

(1) reflexive - $x \leq x$ for $x \in S$
(2) transitive - if $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in S$
(3) anti-symmetric - for $x, y \in S$, if $x \leq y$ and $y \leq x$, then $x = y$ (this can also be stated as if x is less than y, then y is not less than x).

If x is not less than or equal to y and y is not less than or equal to x then we say that x and y are not comparable. If is always the case that either x is less than or equal to y or y is less than or equal to x for every x and y in the set then we say that the partial order is a *total order*.

Almost any set where you order the elements in some way is a partial order. If you have a way of deciding how every element compares to every other element then this is called a total order on a set (and it is still a partial order). However in most partial orders there are some elements which are not comparable.

Example 1. The first simple example that we are aware of just the positive integers with the relation $x \leq y$ representing "x is less than or equal to y." This is an example of a partial order which is a total order.

Example 2. The second example is the set of positive integers where the relation is "x divides y". We usually denote this relation $a|b$ to mean that there exists a k such that $ak = b$. Note that $a|b$ is either true or false, it is not a number. Any positive integer divides itself (so this relation is reflexive), if x divides y and y divides z then x divides z (so it is transitive). Also if x divides y and y divides x then x and y are equal (anti-symmetric).

This is an example of a partial order which is not a total order because for instance 3 does not divide 5 and 5 does not divide 3 so they are not comparable in this partial order.

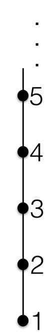
Example 3. The third example we take as our set the set of subsets of $\{1, 2, \ldots, n\}$ where n is a fixed integer (which is often denoted B_n). This example is slightly different than the previous two since for a fixed n it has a finite number of elements. A subset S will be less than a subset T if S is contained in T. Again we can check that this relation is reflexive, transitive and anti-symmetric. Since the empty set is a subset of every subset it is a smallest element of the set.

This is another example of a partial order which is not a total order because, for instance, the sets $\{1, 2\}$ and $\{2, 4, 6\}$ would not be comparable.

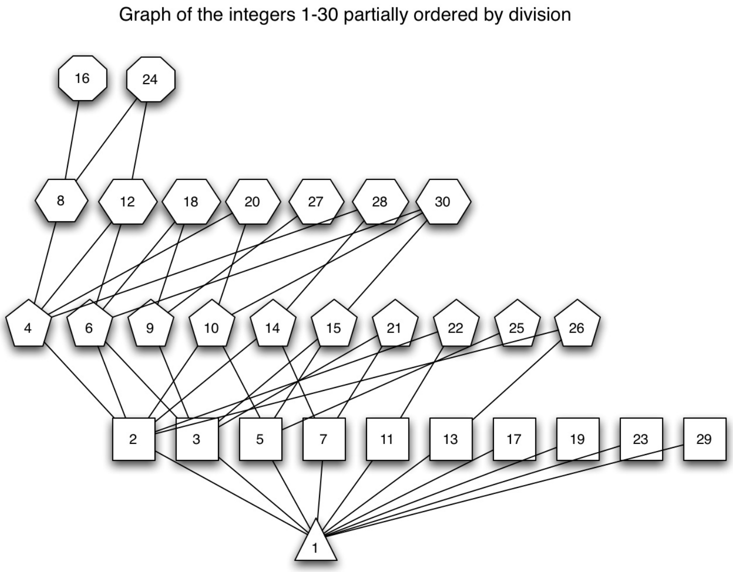
Example 4. As a fourth example I considered the set partitions of $\{1, 2, \ldots, n\}$ where n is again a fixed integer. We say that a set partition $\{S_1, S_2, \ldots, S_k\}$ is smaller than or equal to a set partition $\{T_1, T_2, \ldots, T_\ell\}$ if each set S_i is a subset a set T_j.

These partial orders can be pictured by drawing a dot for each element of the set and placing the smallest elements at the bottom of the picture and the larger elements above. Then we draw a line between x and y if $x \leq y$ and if there are no other elements z such that $x \leq z \leq y$. This is called the Hasse diagram of the poset.

Example 5. The diagram for the first few integers ordered by the usual less than or equal to order looks like a vertical line.

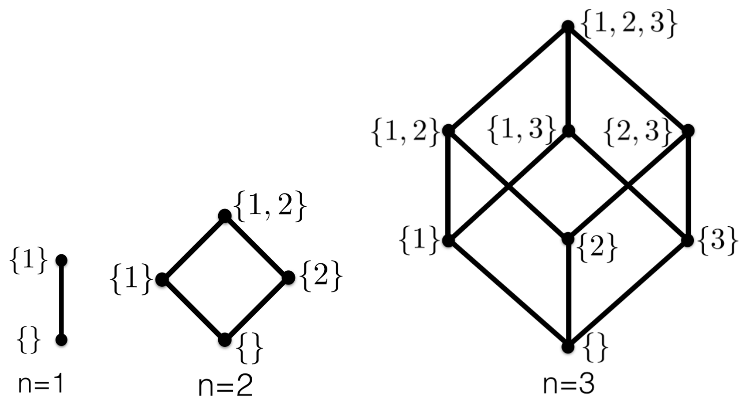


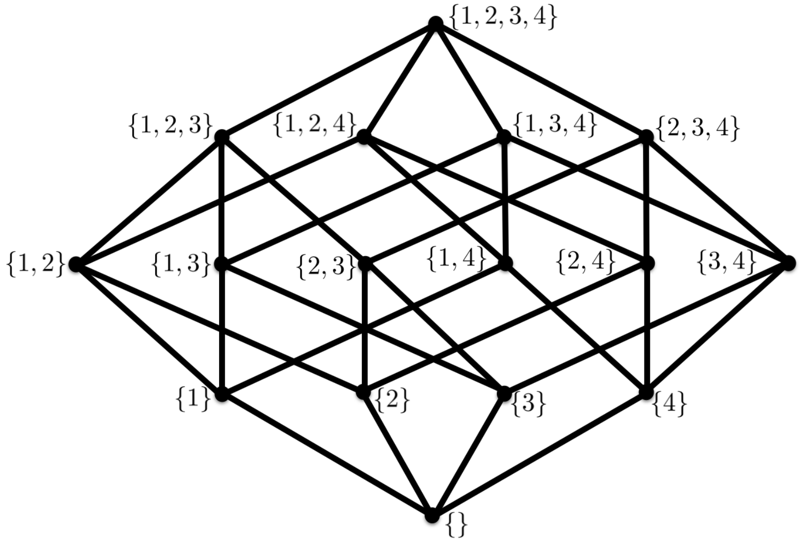
Example 6. The diagram for the first 30 integers ordered by division is the diagram



Notice that 1 is on the first level, the second level consists of the prime numbers, the third row consists of integers which are the product of two primes, the fourth level is the product of three primes, etc.

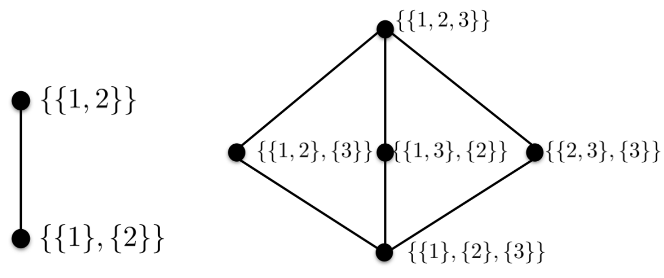
Example 7. The diagram for the poset of subsets of $\{1, 2, \ldots, n\}$ has the following diagrams for $n = 1, 2, 3, 4$.



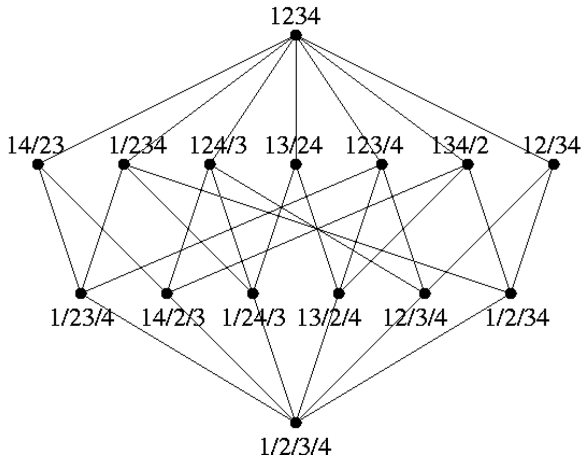


n=4

Example 8. For the partial order of set partitions defined in example **??** for $n = 2, 3, 4$ (for $n = 1$ there is just one element) the diagram for this partial order is the following three pictures.



n=2 n=3



$n = 4$

Note that the last image was taken from
`http://math.berkeley.edu/~lpachter/249_Spring_2009/reading/index.html`.

The set partitions are represented with the numbers that are in the same set in a group and the / indicates that the the next numbers are in a different set so that for instance the set partition $\{\{1,4\},\{2,3\}\}$ is represented by 14/23. The order of the groups and the numbers within the groups don't really matter.

With some minimal conditions on a partially ordered set we have the following theorem.

Theorem 9. *Let $(S, \leq)$ be a partially ordered set such that the number of elements less than or equal to any element x is finite, then given sequences a_x and b_x which are real numbers for each $x \in S$, there exists a real valued function μ on $S \times S$ such that*

$$a_x = \sum_{y \leq x} b_y$$

if and only if

$$b_x = \sum_{y \leq x} \mu(x, y) a_y.$$

Now for each poset it is an interesting problem to find what the function $\mu(x, y)$ is. I gave you the formula for three of these functions μ. One of them we had seen before.

Proposition 10. *(Telescoping sums) For n an integer,*

$$a_n = \sum_{i=1}^{n} b_i,$$

if and only if

$$b_n = a_n - a_{n-1} \ .$$

that is,

$$\mu(n,k) = \begin{cases} 1 & \text{if } k = n \\ -1 & \text{if } k = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 11. *(usual Möbius inversion) For n an integer,*

$$a_n = \sum_{d|n} b_d,$$

if and only if

$$b_n = \sum_{d|n} \mu(n,d) a_d$$

where,

$$\mu(n,d) = \begin{cases} (-1)^k & \text{if } n/d = p_1 p_2 \cdots p_k \text{ where all the } p_i \text{ are distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 12. *(Inclusion-Exclusion) For S a subset of the integers $\{1, 2, \ldots, n\}$,*

$$a_S = \sum_{T \subseteq S} b_T,$$

if and only if

$$b_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} a_T,$$

that is,

$$\mu(S,T) = \begin{cases} (-1)^{|S|-|T|} & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

When I mentioned that we had already seen one of these I was referring to Proposition **??**. We used this in the first few days of class and it is called the method of telescoping sums. Look back at the notes from some of the first classes from this year.

____________________ From September 11, 2014 ____________________

In order to show that

$$b(1) + b(2) + \cdots + b(n) = a(n)$$

for some formulas $a(n)$ and $b(n)$ and $b(0) = 0$, then all you need to do is show that $a(n) - a(n-1) = b(n)$. If you do then

$$a(n) - a(n-1) = b(n)$$
$$a(n-1) - a(n-2) = b(n-1)$$
$$\vdots$$
$$a(2) - a(1) = b(2)$$

$$a(1) - a(0) = b(1)$$

Now add up all the terms on the left hand side and we have

$$(a(n) - a(n-1)) + (a(n-1) - a(n-2)) + \cdots + (a(1) - a(0)) = a(n) - a(0) = a(n)$$

If you add up all the terms on the right hand side of the equality then you have

$$b(n) + b(n-1) + \cdots + b(2) + b(1)$$

and they must be equal.

(For point of clarification, I exchanged a and b in this excerpt to ensure that they agree my notation here)

Lets do the same thing for the Proposition **??**. In this case the first 8 terms of our sequence are

$$\begin{aligned}
a_1 &= b_1\\
a_2 &= b_2 + b_1\\
a_3 &= b_3 + b_1\\
a_4 &= b_4 + b_2 + b_1\\
a_5 &= b_5 + b_1\\
a_6 &= b_6 + b_3 + b_2 + b_1\\
a_7 &= b_7 + b_1\\
a_8 &= b_8 + b_4 + b_2 + b_1\\
&\vdots
\end{aligned}$$

Now if we start by solving for the b-sequence in terms of the a-sequence the we see

$$\begin{aligned}
b_1 &= a_1\\
b_2 &= a_2 - a_1\\
b_3 &= a_3 - a_1\\
b_4 &= a_4 - a_2\\
b_5 &= a_5 - a_1\\
b_6 &= a_6 - a_3 - a_2 - a_1\\
b_7 &= a_7 - a_1\\
b_8 &= a_8 - a_4\\
&\vdots
\end{aligned}$$

It looks the same as in the case of $a_n = \sum_{i=1}^n b_n$ except that in the case of b_6 there are 4 terms. It is rarely the case the case that there is just one exception. In fact, Proposition

?? is telling us that there will be 2^k terms in the expression for b_n if n is a product of k distinct primes. For example if we solve for b_{36}, because $36 = 2^2 \cdot 3^2$ we have that

$$b_{36} = a_{36} - a_{18} - a_{12} + a_6$$

while since $30 = 2 \cdot 3 \cdot 5$ we will have

$$b_{30} = a_{30} - a_{15} - a_{10} - a_6 + a_5 + a_3 + a_2 - a_1 \ .$$

For each of these propositions, there is something to prove. We have more or less shown Proposition **??**. The ones that are slightly more difficult to show are Propositions **??** and **??**.

1. Inclusion-Exclusion

Proposition **??** is the easier of the two to prove (I will include the proofs below) and it can be used to solve a class of counting problems that are elementary but difficult (remember the definition of 'elementary' is that it is easy to state with very little background knowledge, not that it is easy to solve). Before I provide the justification, let me show you the application.

Example 13. I went to a textbook on combinatorics and looked that there was a whole section of problems on inclusion-exclusion. I just picked out the first problem I could find:

How many 5 card hands have at least one card from each suit?

Let S be a set of suits

$$A_S = \text{ \# of 5 card hands with suits from } S \text{ (some suits might not be included)}$$

and

$$B_S = \text{ \# of 5 card hands with suits exactly from } S \text{ (each suit appears at least once)}$$

then these sequences of numbers are related by

$$A_S = \sum_{T \subseteq S} B_T$$

because of the addition principle. Every 5 card hand has a unique set of suits which appear in the hand T and that hand will be counted in the count from B_T. Hence (by Möbius inversion/inclusion-exclusion) they are also related by

$$B_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} A_T \ .$$

The thing is we can count the number of hands in A_S. If $S = \{\heartsuit\}$ or $S = \{\clubsuit\}$ or $S = \{\spadesuit\}$ or $S = \{\diamondsuit\}$, then $A_S = A_{\{\heartsuit\}} = \binom{13}{5}$. If S has two suits (e.g. $S = \{\heartsuit, \spadesuit\}$ and there are 6 sets like this) then $A_S = \binom{26}{5}$. If S contains three suits (e.g. $S = \{\heartsuit, \spadesuit, \diamondsuit\}$

and there are 4 sets like this) then $A_S = \binom{39}{5}$. Finally $A_{\{\heartsuit,\spadesuit,\clubsuit,\diamondsuit\}} = \binom{52}{5}$. In case it needs to be said, $A_{\{\}} = 0$ because there are no 5 card hands that don't use any suits. Therefore,

$$B_{\{\heartsuit,\spadesuit,\clubsuit,\diamondsuit\}} = \binom{52}{5} - 4\binom{39}{5} + 6\binom{26}{5} - 4\binom{13}{5}$$

and this is equal to the number of 5 card hands which contain at least one card of each suit.

If you look up inclusion-exclusion, it is usually stated as the identity which generalizes

$$|S \cup T| = |S| + |T| - |S \cap T|$$

and

$$|S \cup T \cup R| = |S| + |T| + |R| - |S \cap T| - |S \cap R| - |T \cap R| + |S \cap T \cap R|$$

and then more generally the formula for the number of elements in the union of k sets is

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{\{\} \neq J\{1,2,\ldots,n\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} S_j \right| .$$

Since this formula is not obviously like the one I stated before, I should try to explain how they are equivalent. Let

$$A_J = \#\text{ of elements in } \bigcup_{j \in J} S_j$$

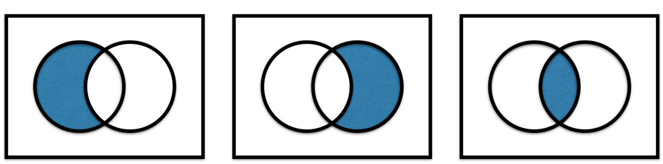
and

$$B_J = \#\text{ of elements in } \bigcap_{j \in J} S_j \text{ but not in any } S_j \text{ for any } j \notin J$$

Then the numbers B_J represent the numbers of elements that are in disjoint sets and so by the addition principle, the two sequences are related by

$$A_J = \sum_{I \subseteq J} B_I \qquad B_J = \sum_{I \subseteq J} (-1)^{|J|-|I|} A_I$$

For example $B_{\{1\}}$ is equal to the number of elements in S_1 which are not also in another set S_i for $i \neq 1$. and $B_{\{1,2\}}$ represents the number of elements in S_1 and S_2 which are not in another S_i for $i \neq 1, 2$. In the case there are only two sets S_1 and S_2 then $B_{\{1\}}$, $B_{\{2\}}$ and $B_{\{1,2\}}$ represents the elements by the following Venn diagrams.



$B_{\{1\}}$ $\qquad B_{\{2\}}$ $\qquad B_{\{1,2\}}$

Now

$$A_{\{1,2\}} = B_{\{1\}} + B_{\{2\}} + B_{\{1,2\}}$$

and

$$B_{\{1,2\}} = A_{\{1\}} + A_{\{2\}} + A_{\{1,2\}}$$