CHAPTER 2: COUNTING FINITE SETS, MAINLY BY RECURSION

MIKE ZABROCKI - 2017 VERSION

1. Some basic principles

1. *The equality principle* - If there exists a bijection between two sets A and B then $|A| = |B|$ (note that $|A|$ is the symbol I will use for the number of elements in the set A).

Example 1. Consider "the set of subsets of $\{1, 2, \ldots, n\}$ that contain both 1 and n" and "the set of subsets of $\{1, 2, \ldots, n-2\}$. I claim that there is a bijection between both of these two sets because if we take $\{1, n, a_1, a_2, \ldots, a_k\}$ as a set which contains both 1 and n and the $2 \leq a_i \leq n-1$ then this is sent to set $\{a_1 - 1, a_2 - 1, \ldots, a_k - 1\}$ is a subset of $\{1, 2, \ldots, n-2\}$.

In the case of $n = 4$ we see that there are four subsets which contain $\{1, 4\}$, namely $\{\{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}$. There are also four subsets of $\{1, 2\}$, namely $\{\{\}$, $\{1\}$, $\{2\}$, $\{1, 2\}\}$.

I will often not state that I am using this principle, just that one description is equal to another and then so "clearly" these two sets have the same number of elements are equal. The word *clearly* is loaded. What it means is that there is some detail to be understood at this point and you need to figure it out (and hence it probably isn't clear in any way).

2. *The addition principle* - If there are three sets related by $A = B \uplus C$ (which means A is the union of B and C and both B and C don't have elements in common), then $|A| = |B| + |C|$.

Example 2. Lets define $\binom{n}{k}$ to be a symbol which represents the number of subsets of an n element set with exactly k elements. So, for instance $\binom{4}{2} = |\{\ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}|$ Consider $n \geq k \geq 1$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. Every k element subset of the set $\{1, 2, \ldots, n\}$ either contains n or it does not. If the set contains n, then it has $k-1$ other elements from $\{1, 2, \ldots, n-1\}$. Otherwise, the set is has k elements from $\{1, 2, \ldots, n-1\}$. □

3. *The multiplication principle* - If the set A consists of all pairs (x, y) where x is an element of B and y is an element of C, then $|A| = |B| \cdot |C|$.

Example 3. Again consider the case when $n \geq k \geq 1$, then

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$$

Proof. The left hand side of this equation represents the number of pairs consisting of a subset of $\{1, 2, \ldots, n\}$ of with k elements and a choice of one of the k elements which will be colored orange.

The right hand side consists of all pairs whose first element is one of the numbers a where $1 \leq a \leq n$ to painted orange, followed by a $k-1$ element subset of $\{1, 2, \ldots, n\} \backslash \{a\}$. $\square$

That implies for $n \geq 1$ and $n \geq k \geq 1$,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!(n-k)!} .$$

2. Sets, set partitions, multisets

A just as a set is a collection of elements without repetition and considered without order, we say that a multiset is a collection of elements with repetition and considered without order. We indicate that the collection is a set by enclosing the elements in $\{$ and $\}$. For a multiset we will use the notation $\{\!\{$ and $\}\!\}$ to enclose the elements. For example a set $\{1, 3, 4, 6, 7\}$ is a set of integers and $\{\!\{1, 1, 1, 2, 3, 4, 4, 7, 8, 8\}\!\}$ is a multiset of integers.

One way of representing the multiset, S, is to consider the multiplicities that an element appears in S. If $x_1, x_2, \ldots, x_r$ are the distinct elements that appear in S, and x_i appears in S with multiplicity a_i, then we will represent $S = \{\!\{x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}\}\!\}$ where we may potentially have an $a_i = 0$ to mean that there are no occurrences of x_i in S. The *size* of a multiset is the number of elements in the multiset and is equal $a_1 + a_2 + \cdots + a_r$.

Let $S = \{\!\{x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}\}\!\}$ and $T = \{\!\{x_1^{b_1}, x_2^{b_2}, \ldots, x_r^{b_r}\}\!\}$ be two multisets. Just as union and intersection are defined on sets, we can define analogous operations on multisets. The union operation will be denoted by $\uplus$ and it is defined as

$$S \uplus T = \{\!\{x_1^{a_1+b_1}, x_2^{a_2+b_2}, \ldots, x_r^{a_r+b_r}\}\!\}$$

while intersection is

$$S \cap T = \{\!\{x_1^{min(a_1,b_1)}, x_2^{min(a_2,b_2)}, \ldots, x_r^{min(a_r,b_r)}\}\!\} .$$

Note that these operations make sense even if either S or T are sets, but the result of the $\uplus$ operation will in general be a multiset and if either S or T is a set then the result of the $\cap$ operation will be a set.

Example 4. Consider the multisets $S = \{1, 3, 4, 6, 7\}$ and $T = \{\!\{1, 1, 1, 2, 3, 4, 4, 7, 8, 8\}\!\}$, the operation

$$S \uplus T = \{\!\{1, 1, 1, 1, 2, 3, 3, 4, 4, 4, 7, 7, 8, 8\}\!\} = \{\!\{1^4, 2, 3^2, 4^3, 7^2, 8^2\}\!\}$$

and

$$S \cap T = \{1, 3, 4, 7\}$$

Example 5. Let $\left(\!\!\binom{n}{k}\!\!\right)$ represent the number of multisets containing the integers $\{1, 2, \ldots, n\}$ element set of size k. For $n \geq 1$ and $k \geq 1$,

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n}{k-1}\!\!\right) + \left(\!\!\binom{n-1}{k}\!\!\right) .$$

Proof. We note that each multiset S in this collection $\{\!\{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}\!\}$ where $a_1 + a_2 + \cdots + a_n = k$. The set of multisets of the integers 1 through n with k elements is the disjoint union of those that contain at least one n and those that do not contain n.

The multisets of n with k elements which contain at least one n are in bijection with the multisets of n with $k-1$ elements (and hence there are $\left(\!\!\binom{n}{k-1}\!\!\right)$ of them) by deleting a value of n from the multiset.

The multisets of n with k elements which do not contain an n are equal to the multisets of $n-1$ with k elements (and hence there are $\left(\!\!\binom{n-1}{k}\!\!\right)$ of them).

The result holds by the addition principle. □

Example 6. The symbol $\left(\!\!\binom{n}{k}\!\!\right)$ representing the number of multisets of an n element set of size k is equal to $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

Proof. There is a bijection between the multisets of an n element set of size k and the sequences of $\bullet$ and $|$ where the multiset $\{\!\{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}\!\}$ is represented by (are in bijection with) the words of the form

$$\underbrace{\bullet\bullet\cdots\bullet}_{a_1\ times}|\underbrace{\bullet\bullet\cdots\bullet}_{a_2\ times}|\cdots|\underbrace{\bullet\bullet\cdots\bullet}_{a_n\ times} .$$

This is equal to the number of sequences of k "$\bullet$" symbols and $n-1$ "$|$" symbols and there are in total $n+k-1$ positions. Each of these words are also represented by a subset of the numbers 1 through $n+k-1$ of size $n-1$ representing the positions of the $|$ symbols. □

Now a set partition of the set $\{1, 2, \ldots, n\}$ is a collection of non-empty, disjoint subsets $A = \{S_1, S_2, \ldots, S_k\}$ such that $S_1 \cup S_2 \cup \cdots \cup S_k = \{1, 2, \ldots, n\}$. They are considered without regard to order. The sets, S_i, are called the *parts* of the partition. The integer k is sometimes referred to as the the *length* of the set partition.

Similarly, a multiset partition of a multiset $\{\!\{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}\!\}$ is a collection of non-empty multisets, $\pi = \{\!\{S_1, S_2, \ldots, S_k\}\!\}$ but now the multisets might not be disjoint and there might be repeats of the multisets in the collection. We again refer to the S_i as the *parts* of the multiset partition and k as the *length*.

Somewhere in between these two is a set partition of a multiset $\{\!\{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}\!\}$. These will be collections of non-empty sets $\pi = \{\!\{S_1, S_2, \ldots, S_k\}\!\}$ whose parts S_i are sets and the elements do not have repetition.

Example 7. The set partitions of $\{1, 2, 3\}$ are

$$\{\{1, 2, 3\}\}$$

$$\{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2, 3\}\}$$

$$\{\{1\}, \{2\}, \{3\}\}$$

and the set partitions of $\{1, 2, 3, 4\}$

$$\{\{1, 2, 3, 4\}\}$$

$$\{\{1, 2, 3\}, \{4\}\}, \{\{1, 2, 4\}, \{3\}\}, \{\{1, 3, 4\}, \{2\}\}$$
$$\{\{2, 3, 4\}, \{1\}\}, \{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}$$

$$\{\{1, 2\}, \{3\}, \{4\}\}, \{\{1, 3\}, \{2\}, \{4\}\}, \{\{1, 4\}, \{2\}, \{3\}\},$$
$$\{\{2, 3\}, \{1\}, \{4\}\}, \{\{2, 4\}, \{1\}, \{3\}\}, \{\{3, 4\}, \{1\}, \{2\}\}$$

$$\{\{1\}, \{2\}, \{3\}, \{4\}\}$$

If we make a table of the numbers of these objects organizing in rows by the value of n and the columns by the number of parts (and filling in for $n = 1, 2$), we see

1			
1	1		
1	3	1	
1	7	6	1

which coincides with the coefficients $\left\{ n \atop k \right\}$ of $(x)_k$ in the expansion of x^n.

3. Permutations

A permutation of $\{1, 2, \ldots, n\}$ (or sometimes just "a permutation of n") is a map from $\{1, 2, \ldots, n\}$ to itself which is one-to-one and onto (a bijection). It can also be thought of as an ordering of the numbers of 1 through n or a list of the numbers 1 through n, each used once without repetition.

We typically use 3 different notations/ways of representing permutations and we will switch between them as needed.

If we thinking of a permutation of the ordering of the integers $\{1, 2, \ldots, n\}$ without repetition that we usually represent this as a list not separated by commas (mainly because it is prettier and saves space). For instance one example of permutation of 9 would be represented as

$$428365179 \ .$$

We call this the one-line notation for the permutation, but it is identified with the map which sends 1 to 4, 2 to 2, 3 to 8, 4 to 3, 5 to 6, 6 to 5, 7 to 1, 8 to 7 and 9 to 9. If we call that permutation a function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, then $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 8$, $\sigma(4) = 3$, $\sigma(5) = 6$, $\sigma(6) = 5$, $\sigma(7) = 1$, $\sigma(8) = 7$, $\sigma(9) = 9$. But then as a second notation we could represent this by the diagram

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 8 & 3 & 6 & 5 & 1 & 7 & 9 \end{pmatrix} .$$

This second way of representing the permutation is called two-line notation. The disadvantage of this notation over that of one line notation is that it is bulkier, but it more clearly indicates that we are thinking of the permutation as a map which we can compose.

The third way that we use to represent permutations is called *cycle notation.* A sequence of integers of the form $(i_1 i_2 \cdots i_r)$ represents i_1 is sent to i_2, i_2 is sent to i_3, and so on until i_{r-1} is sent to i_r and i_r is sent to i_1. A cycle of the form (j_1) means that j_1 is sent to j_1. Every permutation can be represented as a product of disjoint cycles.

We will represent σ in cycle notation, that is write it as

$$\sigma = (i_1, i_2, \ldots, i_{c_1})(j_1, j_2, \ldots, j_{c_2}) \cdots (\ell_1, \ell_2, \ldots, \ell_{c_r})$$

where the integers $\{1, 2, \ldots, n\}$ appear exactly once in the permutation. This notation means
$\sigma(i_k) = i_{k+1}$ for $1 \leq k < c_1$ and $\sigma(i_{c_1}) = i_1$
$\sigma(j_k) = j_{k+1}$ for $1 \leq k < c_2$ and $\sigma(j_{c_2}) = j_1$
$\vdots$
$\sigma(\ell_k) = \ell_{k+1}$ for $1 \leq k < c_r$ and $\sigma(\ell_{c_1}) = \ell_1$.

Now in our example above we have that in cycle notation it is represented as

$$(14387)(56)(2)(9) .$$

Note that this representation in cycles is not unique in that the cycles can appear in any order (e.g. (2)(14387)(9)(56) is equivalent) and each of the cycles of length longer than 1 are equal if we cyclicly rotate their values (e.g. (65)(38714)(2)(9) is an equally valid way of representing the same permutation by disjoint cycles). For shorter notation one need not include the cycles of length 1 and just assume that i is sent to i if it doesn't appear in the cycle notation for the permutation.

Example 8. Let $x'(n, k)$ be the number of permutation of $\{1, 2, \ldots, n\}$ into k disjoint cycles. Lets list the permutations of $\{1, 2, 3\}$

$$(123), (132), (1)(23), (2)(13), (3)(12), (1)(2)(3)$$

and the permutations of $\{1, 2, 3, 4\}$ listed by the number of cycles is

$$(1234), (1243), (1324), (1342), (1423), (1432)$$

$$(12)(34), (13)(24), (14)(23), (123)(4), (132)(4),$$
$$(124)(3), (142)(3), (134)(2), (143(2), (1)(234), (1)(243)$$

$$(12)(3)(4),(13)(2)(4),(14)(2)(3),(1)(23)(4),(1)(24)(3),(1)(2)(34)$$

$$(1)(2)(3)(4)$$

If we make a table of the numbers of these objects organizing in rows by the value of n and the columns by the number of parts (and filling in for $n = 1, 2$), we see

1			
1	1		
2	3	1	
6	11	6	1

which coincides with the coefficients $s'(n, k)$ where $(-1)^{n+k}s'(n, k)$ is the coefficient of x^k in the expansion of $(x)_n$.

We note that for $n \geq 1$, and for $2 \leq k \leq n$, that for a permutation of $\{1, 2, \ldots, n\}$ into k disjoint cycles it is either the case that n is in a cycle by itself, or n is in another cycle where a is sent to n for some $1 \leq a \leq n - 1$.

The number of permutations of $\{1, 2, \ldots, n\}$ into k disjoint cycles where n is in a cycle by itself is equal to the number of permutations of $\{1, 2, \ldots, n-1\}$ into $k-1$ disjoint cycles because these sets are in bijection by deleting the cycle (n). There are $x'(n - 1, k - 1)$ of these permutations.

By the multiplication principle, the number of permutations of $\{1, 2, \ldots, n\}$ into k disjoint cycles where n follows one of the $n - 1$ other integers is equal to $(n - 1)x'(n - 1, k)$.

We conclude by the addition principle that

$$x'(n, k) = (n - 1)x'(n - 1, k) + x'(n - 1, k - 1) \ .$$

Now for $n \geq 1$, there is exactly one permutation of $\{1, 2, \ldots, n\}$ into n disjoint cycles, namely, $(1)(2) \cdots (n)$, hence $x'(n, n) = 1$.

Also for $n \geq 2$, every permutation of $\{1, 2, \ldots, n\}$ in a single cycle is equal has a is sent to n for some $1 \leq a \leq n$. By deleting the n from the cycle, the multiplication principle says that the number of permutations of $\{1, 2, \ldots, n\}$ in one cycle is $n - 1$ times the number of permutations of $\{1, 2, \ldots, n - 1\}$ in one cycle, or to restate as an equation $x'(n, 1) = (n - 1)x'(n - 1, 1) = (n - 1)!$.

Next we give a quick inductive argument that $s'(n, k) = x'(n, k)$ for $n \geq 1$ and $1 \leq k \leq n$.

Proof. First we note that $x'(n, n) = s'(n, n) = 1$ and $x'(n, 1) = s'(n, 1) = (n-1)!$ for $n \geq 1$. In particular $x'(1, 1) = s'(1, 1) = 1$.

Then assume by induction that $x'(n - 1, k) = s'(n - 1, k)$ for $1 \leq k \leq n - 1$. Then we have just shown that

$$x'(n, k) = (n-1)x'(n-1, k)+x'(n-1, k-1) = (n-1)s'(n-1, k)+s'(n-1, k-1) = s'(n, k)$$

where the middle equality follows from the inductive hypothesis and the last equality follows from the definition of $s'(n, k)$. By the principle of mathematical induction $x'(n, k) = s'(n, k)$ for all $n \geq 1$ and $1 \leq k \leq n$. □

I remark that this example has all the details of an inductive argument which I would like you to understand and follow. Normally however we would just specify that two sequences have the same recurrence and satisfy the same base case, therefore they must be equal (but, of course, you do need to check those details).

4. Summary of notation

The symbols $s'(n,k)$ are the Stirling numbers of the first kind (sometimes references will call these 'the unsigned Stirling numbers of the first kind' to emphasize that the signed Stirling numbers of the first kind are actually $(-1)^{n+k}s'(n,k)$). We showed that they are the number of permutations of $\{1,2,3,\ldots,n\}$ that have k cycles. There are $n!$ permutations of $\{1,2,3,\ldots,n\}$ in total. Hence,

$$n! = \sum_{k=1}^{n} s'(n,k)$$

The symbols $S(n,k) = \left\{ n \atop k \right\}$ are the Stirling numbers of the second kind. We showed (or will show in exercises) that they are the number of set partitions $\{1,2,3,\ldots,n\}$ into k parts. The total number of set partitions of $\{1,2,3,\ldots,n\}$ are called the Bell numbers (we will give at least one formula for the Bell numbers later) and that sequence will be denoted B_n (the first few values are $1,2,5,15,52,203,\ldots$).

$$B_n = \sum_{k=1}^{n} \left\{ n \atop k \right\}$$

Finally we had the binomial coefficients $\binom{n}{k}$ (and for $n \geq 0$ and $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and the multichoose coefficients $\left(\!\!\binom{n}{k}\!\!\right)$. They are related by $\binom{n+k-1}{k} = \left(\!\!\binom{n}{k}\!\!\right)$. The binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in $(x+1)^n$. In summary, they may also be defined by recurrences which make them seem quite similar. For $n \geq 1$ and $1 < k < n$,

$$\begin{aligned}
s'(n,k) &= (n-1)s'(n-1,k) + s'(n-1,k-1) \\
\left\{ n \atop k \right\} &= k \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\} \\
\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\
\left(\!\!\binom{n}{k}\!\!\right) &= \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right) \quad .
\end{aligned}$$

A recurrence alone doesn't define these numbers, we also need to know a base case and here we have $s'(n,n) = \left\{ n \atop n \right\} = \binom{n}{n} = \left\{ n \atop 1 \right\} = \left(\!\!\binom{1}{n}\!\!\right) = 1$ and $s'(n,1) = (n-1)!$, $\binom{n}{1} = \left(\!\!\binom{n}{1}\!\!\right) = n$.