HOMEWORK #1 - MATH 4160

ASSIGNED: TUESDAY, SEPTEMBER 17, 2019
DUE: THURSDAY, OCTOBER 3, 2019

Write your homework solutions neatly and clearly and in LaTeX. Provide full explanations and justify all of your answers.

(1) Prove the following three identities for $n \geq 0$ using both telescoping sums and induction:
   (a) $1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$.
   (b) $1^4 + 2^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.
   (c) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

(2) The Stirling numbers of the second kind are defined as $\left\{\begin{matrix}n\\1\end{matrix}\right\} = \left\{\begin{matrix}n\\n\end{matrix}\right\} = 1$ for $n \geq 0$, $\left\{\begin{matrix}0\\0\end{matrix}\right\} = 1$, $\left\{\begin{matrix}n\\k\end{matrix}\right\} = 0$ if $k > n$ or $k \leq 0$, and for $2 \leq k \leq n-1$,

$$\left\{\begin{matrix}n\\k\end{matrix}\right\} = k \left\{\begin{matrix}n-1\\k\end{matrix}\right\} + \left\{\begin{matrix}n-1\\k-1\end{matrix}\right\} .$$

Prove that

$$x^n = \sum_{k=1}^{n} \left\{\begin{matrix}n\\k\end{matrix}\right\} (x)_k .$$

FYI: to typeset $\left\{\begin{matrix}n\\k\end{matrix}\right\}$ you should use

`\left\{\begin{matrix}n\\k\end{matrix}\right\}`

(3) The (unsigned) Stirling numbers of the first kind are defined as $s'(n,1) = (n-1)!$ for $n \geq 1$, $s'(n,n) = 1$ for $n \geq 0$, $s'(n,k) = 0$ if $k > n$ or $k \leq 0$ and for $2 \leq k \leq n-1$,

$$s'(n,k) = (n-1)s'(n-1,k) + s'(n-1,k-1) .$$

In class, we defined for $k > 0$, $(x)_k := x(x-1)(x-2)\cdots(x-k+1)$ (falling factorial) and there is also the $(x)^{(k)} := x(x+1)(x+2)\cdots(x+k-1)$ (rising factorial). Show that

$$(x)^{(n)} = \sum_{k=1}^{n} s'(n,k)x^k .$$

(4) For $n \geq 0$ and $1 \leq k \leq n$, let $b'(n,k) =$ the number permutations of $\{1,2,\ldots,n\}$ have k cycles.
   (a) Explain why $b'(n,1) = (n-1)!$ and why $b'(n,n) = 1$.
   (b) In 2-3 sentences, explain why for $2 \leq k \leq n$,

$$b'(n,k) = (n-1)b'(n-1,k) + b'(n-1,k-1) .$$

   (c) Use parts (a) and (b) to show that $b'(n,k) = s'(n,k)$ for all $n \geq 1$ and all $1 \leq k \leq n$.

Enumeration problems. The emphasis is on you coming up with a clear explanation of why the answer is true. I more care why the answer is than what the answer is. I'm going to use terms from non-standard poker hands and you should follow Wikipedia: List of poker hands and Wikipedia: Non-standard poker hands as a reference.

(1) A five card poker hand is dealt from a deck with 52 cards (usual 13 ranks and 4 suits). A big/little cat/dog hands are types of no-pair hands defined by their highest and lowest cards (see Wikipedia article). Count the number of big cat, little cat, big dog, little dog hands and explain why they rank above the straight hands but below the flush hands.

(2) A five card poker hand is dealt from a deck with 54 cards (the usual 52 cards) plus two wild cards. The wild cards can be counted as any particular rank, but not any particular suit (so it is impossible to have a flush with a wild card). Every hand will then fall into a category depending on the highest ranking hand that you can make with the cards. Give an argument (similar to those that are in the notes for the standard poker hands with a 52 card deck) to count the number of each type of poker hand. Show that your count for all of the possible hands adds up to $\binom{54}{5}$.

Combinatorial proofs: For the following problems give a combinatorial proof by describing a set that is counted by the left hand side of the equality and a set that is counted by the right hand side of the equality and explaining why these two sets are the same.

(1)
$$m\binom{m}{2} = 3\binom{m}{3} + 2\binom{m}{2}$$

(2)
$$\binom{2n}{n}\binom{2n}{n-1} = \sum_{k=0}^{n}\binom{2n}{k}\binom{2n-k}{n-k}\binom{n}{n-1-k}$$