HOMEWORK #2 - MATH 4160

ASSIGNED: OCT 5, 2019 DUE: OCT 23, 2019

Write your homework solutions clearly and in LaTeX. Provide full explanations and justify all of your answers. You may use the computer to complete calculations, but please describe the calculation you asked the computer to do (provide the input and output in some form).

(1) Let $A(x) = \sum_{n\geq 0} a_n x^n$ be the generating function for the sequence $a_0, a_1, a_2, a_3, \ldots$. Any term in this sequence can be calculated from the generating function by $a_r = \frac{1}{r!}\frac{d^r}{dx^r}A(x)|_{x=0}$ so you may use a_r in your formula for any fixed r. Give a formula in terms of $A(x)$ and a_r (for any fixed $r \geq 0$) for the generating function of the sequence:
  (a) the n^{th} Fibonacci number times the n^{th} term in the sequence: $a_0F_0, a_1F_1, a_2F_2, a_3F_3, \ldots$ where $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, \ldots$ is the Fibonacci sequence.
  (b) first differences: $a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots$.
  (c) delete $a_3 : a_0, a_1, a_2, a_4, a_5, a_6, a_7, \ldots$
  (d) sum of adjacent terms: $a_0 + a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots$.

  Hint: For part (a) will use the expression that I derived for the Fibbonaci numbers involving the golden ratio.

(2) In class I said how to prove (its in the notes too) that $B_k(x) = \sum_{m\geq 0}\binom{m}{k}x^m = \frac{x^k}{(1-x)^{k+1}}$.
  (a) Starting with the formula for $B_k(x)$, derive an equation for $A_k(x) = \sum_{m\geq 0} m\binom{m}{k}x^m$
  (b) Show that $A_k(x) = kB_k(x) + (k+1)B_{k+1}(x)$
  (c) Derive an identity by taking the coefficient of x^n on both sides of the equation.
  (d) Give a combinatorial argument which proves the same identity.

(3) As in the last problem let $B_k(x) = \sum_{m\geq 0}\binom{m}{k}x^m = \frac{x^k}{(1-x)^{k+1}}$.
  (a) Give an expression for the generating function $A(x, z) = \sum_{k\geq 0}\sum_{m\geq 0}\binom{m}{k}x^m z^k = \sum_{k\geq 0} B_k(x)z^k$.
  (b) Show that by setting $z = 1$, $A(x, 1) = \frac{1}{1-2x} = \sum_{k\geq 0}\sum_{m\geq 0}\binom{m}{k}x^m$. Find an identity by taking the coefficient of x^n on both sides of the equation.
  (c) Show that $A(3x, \frac{-x}{3}) = \frac{1}{1-3x+x^2} = \sum_{k\geq 0}\sum_{m\geq 0}\binom{m}{k}(3x)^m(-x/3)^k$. Since the coefficient of x^n in $\frac{1}{1-3x+x^2}$ is F_{2n+1}, show how the generating function can be used to give a formula for F_{2n+1} in terms of binomial coefficients and powers of 3. What does your formula say in particular about F_9?

(4) Let $G(x) = \sum_{n\geq 0} F_n^2 x^n = \frac{1-x}{(1+x)(1-3x+x^2)}$. Write down an algebraic expression for the generating function of

$$F_{n+3}^2 + F_n^2$$

and another for the generating function of

$$2(F_{n+1}^2 + F_{n+2}^2)$$

and show that they are equal.

(5) Find the generating function for the ways of making change for n cents using nickles, dimes and quarters satisfying the following conditions:
  (a) no restriction
  (b) there are at most 10 nickles and 10 dimes
  (c) there are at most 20 nickles and dimes (together)
  (d) the number of nickles and dimes (together) is even
  (e) the number of coins (nickles, dimes and quarters) is even

  If you want to check your answer on the computer to that you have the correct generating function then you should find that to make change for \$13.55 for condition (a) there are 3781 ways; (b) there are 24 ways; (c) there are 46 ways; (d) there are 1890 ways; for condition (e) there are 1877 ways.