HOMEWORK #3 - MATH 4160

ASSIGNED: OCT 29, 2019 DUE: NOV 13, 2019

LaTeX your homework solutions neatly and clearly. Provide full explanations and justify all of your answers.

(1) Give a proof of the following identity by counting two sets of partitions in two different ways.

$$\prod_{i\geq 1} \frac{1}{1-x^i} = \frac{1}{1-x} + \sum_{r\geq 1} \frac{x^{r(r+1)}}{1-x^{r+1}} \prod_{j=1}^{r} \frac{1}{(1-x^j)^2} \ .$$

Hint: Look for the largest $r \times (r+1)$ rectangle in the partition.

(2) Find the exponential generating function for the number of words of length n using letters a, b, c, d such that

(a) all 4 letters occur without restriction

(b) at least three a's occur and at least four b's occur and the total number of a's and c's is even.

(c) the letters a, b each appear an even number of times and c, d together appear an even number of times

(d) the letters a, b together appear even number of times and a, c, d together appear an even number of times

Use your generating function to find the number of number of words of length 20 with the restrictions above.

(3) Show that the exponential generating function for the sequence $1, 2, 3, 4, \ldots$ is $(1 + x)e^x$. What is the sequence corresponding to the exponential generating function $\frac{1}{1+x}$? Take the coefficient of $\frac{x^n}{n!}$ in the expression

$$\frac{1}{1+x} \cdot (1 + x)e^x = e^x$$

(using the multiplication principle of exponential generating functions on the left hand side) and state the combinatorial identity proved by taking this coefficient.

(4) Let $B'(n, k)$ be the coefficient of x^k in $(x + 1)(x + 3) \cdots (x + (2n - 1))$. Find a formula for the generating function

$$\sum_{n\geq 0} \sum_{k\geq 0} B'(n, k) q^k \frac{x^n}{n!} = e^{-\frac{1}{2}(1+q) log(1-2x)}$$