MIDTERM - TAKE HOME - MATH 4160

ASSIGNED: OCTOBER 31, 2019 DUE: NOVEMBER 4, 2019 AT 11:59PM

Write your solutions neatly and clearly. Provide full explanations and justify all of your answers. DO NOT DISCUSS THESE PROBLEMS WITH OTHERS. You must do this work alone and I will ask you to sign the statement below which states that you have not discussed these problems with others or received help on these problems (when you hand the paper to me). Note that in certain circumstances I am giving you the answer, and it is your job *explain* it. This means that you should write grammatically correct sentences, tell me why two things are equal, and make your calculations clear and easy to follow.

If you have any questions about the problems you may e-mail me at `zabrocki@mathstat.yorku.ca`.

1 (8 pts)	.
2 (10 pts)	
3 (7 pts)	
total(25 pts)	

(1) Use generating functions to determine the number of ways of putting 100 red, blue and green balls into two bins such that there are an even number of red and blue balls in the first bin and there are at most 30 green balls.

(2) Define for $n \geq 1$, $(x)^{\{n\}} = (x+1)(x+3)\cdots(x+(2n-1))$. Let $B'(n,k)$ be the coefficient of x^k in $(x)^{\{n\}}$, that is,

$$(x)^{\{n\}} = \sum_{k=0}^{n} B'(n,k)x^k \ .$$

Recall that a cycle of a permutation π is an ordered list of numbers $(a_1, a_2, \ldots, a_r)$ such that $\pi(a_i) = a_{i+1}$ for $1 \leq i < r$ and $\pi(a_r) = a_1$ (with a_1 the the smallest entry in the cycle so that the representation of the cycle is unique). In the homework I asked you to show that the coefficient x^k in $x(x+1)(x+2)\cdots(x+(n-1))$ was equal to the number of permutations π of $\{1, 2, \ldots, n\}$ with exactly k cycles. I'm going to ask you to prove the analogous interpretation for $B'(n,k)$.

(a) (1 pt) Make a table of $B'(n,k)$ for $1 \leq n \leq 3$ and $0 \leq k \leq n$.

(b) (1 pt) Prove that $\sum_{k=0}^{n} B'(n,k) = 2^n n!$ = the number of pairs (S, π) where S is a subset of $\{1, 2, \ldots, n\}$ and π is a permutation of $\{1, 2, \ldots, n\}$.[1]

(c) (1 pt) Give an explicit formula for $B'(n,0)$ and $B'(n,n)$ for all $n \geq 1$.

(d) (2 pts) Give a recurrence for $B'(n,k)$ for $n \geq 2$ and $0 < k < n$

[1] note that I'm asking you to show two equalities in this problem

(e) (5 pts) We will say that a list of numbers $(a_1, a_2, \ldots, a_r)$ is an even cycle of (S, π) if $(a_1, a_2, \ldots, a_r)$ is a cycle of π and the number of elements in $\{a_1, a_2, \ldots, a_r\} \cap S$ is even. Show that $B'(n, k)$ is equal to the number of pairs (S, π) with exactly k even cycles.[2]

(3) Prove the following identity using a combinatorial argument

$$\binom{m}{2}^2 = 6\binom{m}{4} + 6\binom{m}{3} + \binom{m}{2}$$

WHEN YOU SUBMIT THIS EXAM please sign the following statement and fill out the information below.

I attest that I have completed this exam myself without help from anyone else (besides the professor) and I have not discussed the problems on this exam with other students in the class.

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This exam is open book, open notes, and other sources, but I expect you to not ask other people how to complete the assignment. You should list books and websites that you consulted below. Your solutions should refer to precise pages in the notes and reference that I can consult. If you cannot sign the above statement truthfully, I would prefer if you just explain to me the situation rather than perjure yourself. Please detail below the sources you consulted, the that you have obtained on this exam or who you have discussed these problems with:

[2] For example $(\{1\}, (1)(2)(3))$, $(\{2\}, (1)(2)(3))$, $(\{3\}, (1)(2)(3))$, $(\emptyset, (12)(3))$, $(\emptyset, (1)(23))$, $(\emptyset, (2)(13))$, $(\{1,2\}, (12)(3))$ $(\{2,3\}, (1)(23))$, $(\{1,3\}, (2)(13))$ are 9 pairs that have two even cycles

Some people have asked for clarifications on this exam. I will answer them here in case they are useful for everyone.

For "an even number of red and blue balls," do you mean an even number of red and blue balls together, or an even number of red and an even number of blue?

I mean an even number of red and blue balls together.

Is it 100 balls total, of which some are red/blue/green or is it 100 red, 100 blue, 100 green?

100 balls in total.

Should it be: "there are at most 30 green balls in the second bin?"

Its at most 30 balls in total.