F. FRANKLIN’S PROOF OF EULER’S PENTAGONAL NUMBER THEOREM

Abstract. The 18th century mathematician Leonard Euler discovered a simple formula for the expansion of the infinite product $\prod_{i \geq 1} (1 - q^i)$. In 1881, one of the first American mathematicians found an elegant combinatorial proof of this identity.

Proposition 1. (Euler’s pentagonal number theorem)

(1) \[ \prod_{i \geq 1} (1 - q^i) = 1 + \sum_{m \geq 1} (-1)^m \left( q^{m(3m-1)/2} + q^{m(3m+1)/2} \right) \]

There is a clever proof of this proposition that comes from a mathematician F. Franklin [4]. Since this is exactly the sort of proof that is in the spirit of mathematics of algebraic combinatorics it belongs in a course on algebraic combinatorics. Other accounts of this proof can be found in: [5], [6], [7], [8], [9].

Example 1. We note that the left hand side of this equation is the generating function for all strict partitions (partitions where all parts are distinct) weighted with $(-1)^{\ell(\lambda)} q^{|\lambda|}$. That is, \[
(2) \prod_{i \geq 1} (1 - q^i) = \sum_{\lambda \text{ strict}} (-1)^{\ell(\lambda)} q^{|\lambda|}
\]

This follows by observing that to determine the coefficient of $q^n$ by expansion of the product on the left we have a contribution of $(-1)^k q^{\lambda_1 + \lambda_2 + \cdots + \lambda_k}$ for every sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_i > \lambda_{i+1}$ for $1 \leq i < k$. Below we expand the terms of this generating function through degree 8. For example, a term of the form $(-q^4)(-q^2)$ is represented by the picture \[
\begin{array}{cccccccc}
\cdot & \text{ } & \text{ } & \text{ } & \text{ } & \cdot & \text{ } & \text{ }\\
1 & -q & -q^2 & +q^3 & -q^4 & -q^5 & +q^6 & +q^7 \\
\end{array}
\]

and we record the weight of $+q^6$ just below the picture.
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Now we notice that all of the terms cancel except for the ones stated in the theorem, that is we have

$$\prod_{i \geq 1} (1 - q^i) = 1 - q - q^2 + q^5 + q^7 + \cdots$$

In fact, we will show that one way of looking at this expression is to observe terms which survive are those that correspond to the following pictures:

From the image in this example one might think that the theorem would be better named the trapazoidal number theorem. There is a reason that the numbers \( m(3m - 1)/2 \) are referred to as pentagonal numbers and if \( m \to -m \) then the pentagonal number is transformed to \( \to -m(-3m - 1)/2 = m(3m + 1)/2 \). Observe the picture below how a sequence of pentagons have exactly \( m(3m - 1)/2 \) points in them (and this continues for \( m > 3 \)).

Proof. To show that this proposition holds we show that there is an involution \( \phi \) on the strict partitions \( \lambda \) of \( n \) such that \( \phi(\lambda) \) is also a partition of \( n \) and the length of \( \phi(\lambda) \) will have length either one smaller or one larger than that of \( \lambda \). This means that if the weight of a strict partition is \( (-1)^{t(\lambda)} q^{\lambda} \) then the weight of \( \phi(\lambda) \) is \( -(-1)^{t(\lambda)} q^{\lambda} \) and so this term corresponding to \( \phi(\lambda) \) will cancel with the term corresponding to \( \lambda \). This involution will fail to ‘work’ for the partitions of the form \((2m - 1, 2m - 2, \ldots, m)\) which are of size \( 2m^2 - \frac{(m+1)m}{2} = \frac{m(3m-1)}{2} \) and \((2m, 2m - 1, \ldots, m + 1)\) which are of size \( 2m^2 - \frac{(m-1)m}{2} = \frac{m(3m+1)}{2} \).
For a strict partition \( \lambda \) we will let \( r \) equal to the smallest part of \( \lambda \) \((r = \lambda_\ell(\lambda))\) and let \( s \) equal the number of parts which are consecutive at the beginning of the partition. In other words \( s \) is the largest integer such that \((\lambda_1, \lambda_2, \ldots, \lambda_s) = (\lambda_1, \lambda_1 - 1, \ldots, \lambda_1 - s + 1)\).

If \( s \neq \ell(\lambda) \) and \( r > s \) then we will let \( \phi(\lambda) \) equal the partition \((\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_{\ell(\lambda)}, s)\). That is, if the diagram for the partition looks something like the following where there is an \( \times \) in each of the cells corresponding to \( r \) and a dot in the cells corresponding to \( s \)

\[
\begin{array}{ccccccc}
\times & \times & \times & \times & \cdot & \cdot & \cdot \\
\end{array}
\]

then \( \phi(\lambda) \) will be the partition with the diagonal of \( s \) cells filled with a dot moved to the top row of the partition.

\[
\begin{array}{ccccccc}
\cdot & \cdot & \times & \times & \times & \times & \times \\
\end{array}
\]

\( \phi(\lambda) \) has the property that the longest string of consecutive parts at the beginning of the partition is greater than or equal to \( s \).

If \( s \neq \ell(\lambda) \) and \( r \leq s \) then we will let \( \phi(\lambda) \) equal to the partition \((\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{\ell(\lambda)})\). For example, if our diagram is similar to the one below with the cells marked with an \( \times \) representing the row of size \( r \) and those marked with the \( \cdot \) represent the cells which correspond to the \( s \) consecutive parts at the beginning of the partition.

\[
\begin{array}{ccccccc}
\times & \times & \times & \times & \times & \cdot & \cdot \\
\end{array}
\]

The partition corresponding to \( \phi(\lambda) \) is then represented by the following picture.

\[
\begin{array}{ccccccc}
\cdot & \times & \times & \times & \times & \cdot & \cdot \\
\end{array}
\]

Notice that it is also possible that \( s = \ell(\lambda) \). In this case if \( r > s + 1 \) then we will remove the \( s \) cells along the diagonal and turn them into the shortest row so that \( \phi(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, s) \). For example we have the picture on the left will be transformed to the one on the right.

\[
\begin{array}{ccccccc}
\times & \times & \times & \times & \cdot & \cdot & \cdot \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\cdot & \times & \times & \times & \times & \cdot & \cdot \\
\end{array}
\]

If \( s = \ell(\lambda) \) and \( r < s \) then we will set \( \phi(\lambda) = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_{r+1}, \ldots, \ell(\lambda) - 1) \), this corresponds to the case when we have a partition of the form of the one below.
If we describe what is happening to the diagram the map $\phi$ does one of two things, either it removes the smallest row of $r = \lambda_0(\lambda)$ cells of the partition and places one cell more in each of the first $r$ rows (in the case that $r < s$ or $r = s$ and $s < \ell(\lambda)$) or it removes one cell from each of the first $s$ rows and adds a row of size $s$ to the top of the diagram (in the case that $r > s + 1$ or $r = s + 1$ and $s < \ell(\lambda)$).

Observe that if the weight of $\lambda$ is $(-1)^{\ell(\lambda)}$ then since $\phi(\lambda)$ has the same number of cells and either one more or one less row than $\lambda$ then the weight of $\phi(\lambda)$ is the negative of the weight of $\lambda$.

Also observe for each of the 4 cases we have considered, $\phi(\phi(\lambda))$ is just $\lambda$. This implies we can say that in the expansion of the expression $\sum_{\lambda \text{ strict}} (-1)^{\ell(\lambda)} q^{||\lambda||}$, the term corresponding to the partition $\lambda$ will cancel with the term corresponding to the partition $\phi(\lambda)$.

There are two cases that we have not considered. These terms do not cancel. One is that $r = s$ and $s = \ell(\lambda)$ and so we have a partition of the form $(2m-1, 2m-2, \ldots, m)$ and the other is that $r = s + 1$ and $s = \ell(\lambda)$ and this is a partition of the form $(2m, 2m-1, \ldots, m+1)$. □

We encourage the reader to take a pencil and draw an arrow between the diagrams of the strict partitions given in the example above to show that the involution works as expected.

**References**