$p_i \geq 0 \qquad \sum p_i = 1$

$a_i > 0$

Hint: e^x is concave up $\quad F(x) = e^x$

$$p_1 e^{x_1} + p_2 e^{x_2} + \cdots + p_n e^{x_n} = \sum p_i F(x_i)$$

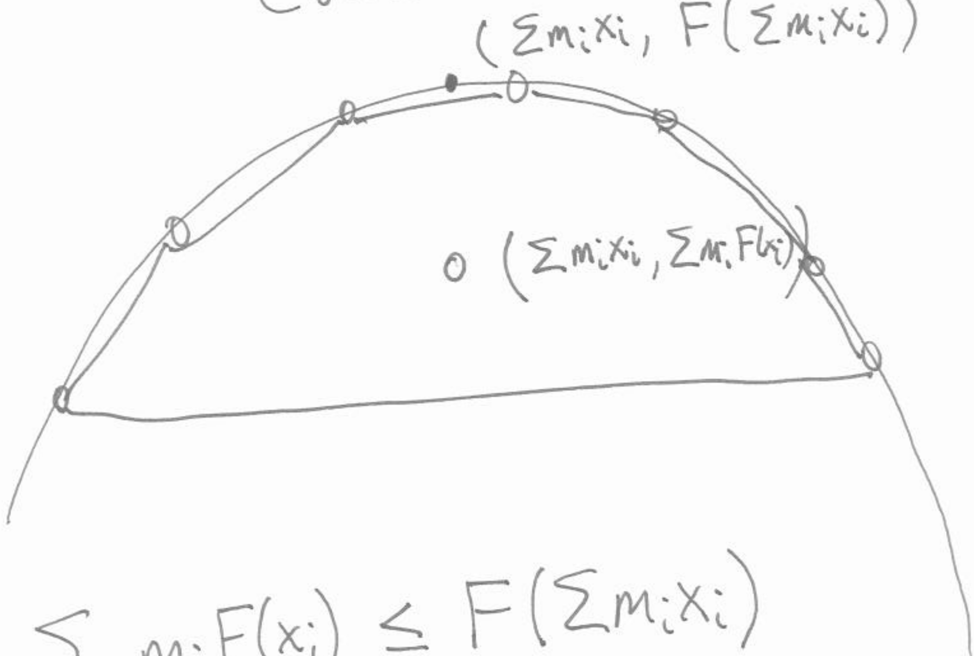
choose $x_i = \log a_i$

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \leq p_1 a_1 + p_2 a_2 + \cdots + p_n a_n$$

$$F\left(\sum m_i x_i\right) = e^{\sum p_i \log a_i} = e^{\sum \log a_i^{p_i}} = e^{\log \prod a_i^{p_i}} = \prod a_i^{p_i}$$

$$H(X|Y) \leq H(X)$$

Concave down function



$$\sum m_i F(x_i) \leq F(\sum m_i x_i)$$

$$\sum m_i = 1 \quad m_i \geq 0$$

$$H(X|Y) = \sum_{b} P[Y=b]\, H(X|Y=b)$$

$$= \sum_{b} P[Y=b] \sum_{a} P[X=a|Y=b] \log_2\left(\frac{1}{P[X=a|Y=b]}\right)$$

$$= \sum_{b} \cancel{P[Y=b]} \sum_{a} \frac{P[X=a \,\&\, Y=b]\, P[X=a]}{P[X=a]\,\cancel{P[Y=b]}} \log_2\left(\frac{1}{P[X=a|Y=b]}\right)$$

$$= \sum_{b} \sum_{a} P[Y=b|X=a]\, P[X=a] \log_2\left(\frac{1}{P[X=a|Y=b]}\right)$$

$$= \sum_{a} P[X=a] \sum_{b} P[Y=b|X=a] \log_2\left(\frac{1}{P[X=a|Y=b]}\right)$$

$$\leq \sum_{a} P[X=a] \log_2\left(\frac{1}{P[X=a]}\right)$$

$$= H(X)$$

$$\Rightarrow H(X|Y) \leq H(X)$$

$F(x) = \log_2(x)$ is concave down.

want

$$\sum_b P[Y=b \mid X=a] \log_2\left(\frac{1}{P[X=a \mid Y=b]}\right)$$

$$= \sum m_i F(x_i)$$

want

$$\log_2\left(\frac{1}{P[X=a]}\right) = F\left(\sum m_i x_i\right)$$

take $m_i = P[Y=b \mid X=a]$

$$x_i = \frac{1}{P[X=a \mid Y=b]}$$

$$\sum_b P[Y=b \mid X=a] \cdot \frac{1}{P[X=a \mid Y=b]} =$$

$$= \sum_b \frac{\cancel{P[Y=b \,\&\, X=a]}}{P[X=a]} \cdot \frac{P[Y=b]}{\cancel{P[Y=b \,\&\, X=a]}}$$

$$= \frac{1}{P[X=a]}$$

Conclusion

$$\log_2\left(\frac{1}{P[X=a]}\right) \geq \sum_b P[Y=b \mid X=a] \log_2\left[\frac{1}{P[X=a \mid Y=b]}\right)$$

If X & Y are independent, $P[X=a|Y=b] = P[X=a]$

$$P[X=a|Y=b] = \frac{P[X=a \,\&\, Y=b]}{P[Y=b]}$$

then

$$H(X|Y) = \sum_b P[Y=b]\, H(X|Y=b)$$

$$= \sum_b P[Y=b] \sum_a P[X=a|Y=b] \log_2\left(\frac{1}{P[X=a|Y=b]}\right)$$

$$= \sum_b P[Y=b] \sum_a P[X=a] \log_2\left(\frac{1}{P[X=a]}\right)$$

$$= \sum_b P[Y=b]\, H(X)$$

$$= H(X)$$

Intuition says:

$$H(X,Y) = H(X) + H(Y|X)$$
$$= H(Y) + H(X|Y)$$

$$H(X,Y) = \sum_{a,b} P[X=a \,\&\, Y=b] \log_2\left(\frac{1}{P[X=a \,\&\, Y=b]}\right)$$

$$= \sum_{a,b} P[X=a] \left(\frac{P[X=a \,\&\, Y=b]}{P[X=a]}\right) \log_2\left(\frac{P[X=a]}{P[X=a]\, P[X=a \,\&\, Y=b]}\right)$$

$$= \sum_{a,b} P[X=a]\, P[Y=b|X=a] \log_2\left(\frac{1}{P[X=a]} \cdot \frac{1}{P[Y=b|X=a]}\right)$$

$$= \sum_{a,b} P[X=a]\, P[Y=b|X=a] \left(\log_2 \frac{1}{P[X=a]} + \log_2 \frac{1}{P[Y=b|X=a]}\right)$$

$$= \sum_{a,b} P[X=a]\, P[Y=b|X=a] \log_2\left(\frac{1}{P[X=a]}\right) + \sum_{a,b} P[X=a]\, P[Y=b|X=a] \log_2\left(\frac{1}{P[Y=b|X=a]}\right)$$

$$= \sum_{a} P[X=a] \log_2\left(\frac{1}{P[X=a]}\right) + \sum_{a} P[X=a] \sum_{b} P[Y=b|X=a] \log_2\left(\frac{1}{P[X=a|Y=b]}\right)$$

$$= H(X) + \sum_{a} P[X=a]\, H(Y|X=a)$$

$$= H(X) + H(Y|X)$$