

WORKSHEET IV: EXAMPLES OF GENERATING FUNCTIONS

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Given that

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + q^5 + \dots$$

Prove the following algebraic equations have the given expansions:

(1)

$$\frac{1-q^k}{1-q} = 1 + q + q^2 + \dots + q^{k-1}$$

(2)

$$\frac{1}{(1-q)^{k+1}} = \sum_{n \geq 0} \binom{n+k}{k} q^n$$

(3)

$$(1+aq)^n = \sum_{k \geq 0} \binom{n}{k} a^k q^k$$

(4)

$$\frac{1+q}{(1-q)^2} = 1 + 3q + 5q^2 + 7q^3 + 9q^4 + \dots = \sum_{n \geq 0} (2n+1)q^n$$

(5)

$$\frac{2}{(1-q)^2} = 2 + 4q + 6q^2 + 8q^3 + 10q^4 + \dots = \sum_{n \geq 0} (2n+2)q^n$$

(6)

$$\frac{q+q^2}{(1-q)^3} = q + 4q^2 + 9q^3 + 16q^4 + 25q^5 + \dots = \sum_{n \geq 0} n^2 q^n$$

(7)

$$\frac{q+4q^2+q^3}{(1-q)^4} = q + 8q^2 + 27q^3 + 64q^4 + 125q^5 + \dots = \sum_{n \geq 0} n^3 q^n$$

(8)

$$-\ln(1-q) = q + q^2/2 + q^3/3 + q^4/4 + q^5/5 + \dots = \sum_{n \geq 1} q^n/n$$

(9)

$$\frac{1}{1-q-q^2} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 8q^5 + 13q^6 + \dots = \sum_{n \geq 0} F_{n+1} q^n$$

(10)

$$\frac{1+2q}{1-q-q^2} = 1 + 3q + 4q^2 + 7q^3 + 11q^4 + 18q^5 + 29q^6 + 47q^7 + \dots = \sum_{n \geq 0} L_{n+1} q^n$$

(11)

$$\frac{1-q}{1-3q+q^2} = 1 + 2q + 5q^2 + 13q^3 + 34q^4 + \cdots = \sum_{n \geq 0} F_{2n+1} q^n$$

(12)

$$\frac{1-q}{(1+q)(1-3q+q^2)} = 1 + q + 4q^2 + 9q^3 + 25q^4 + \cdots = \sum_{n \geq 0} F_{n+1}^2 q^n$$

(13)

$$\frac{1}{(1+q)(1-3q+q^2)} = 1 + 2q + 6q^2 + 15q^3 + 40q^4 + \cdots = \sum_{n \geq 0} F_{n+2} F_{n+1} q^n$$

(14)

$$\frac{2-q}{(1+q)(1-3q+q^2)} = 2 + 3q + 10q^2 + 24q^3 + \cdots = \sum_{n \geq 0} F_{n+3} F_{n+1} q^n$$

(15)

$$\frac{1}{1-3q-q^2} = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n \geq 0} F_{n+1} L_{n+1} q^n = \sum_{n \geq 0} F_{2n+2} q^n$$

(16)

$$\frac{1+4q-2q^2}{(1+q)(1-3q+q^2)} = 1 + 6q + 12q^2 + 35q^3 + 88q^4 + \cdots = \sum_{n \geq 0} F_{n+2} L_{n+1} q^n$$

(17)

$$\frac{1}{\sqrt{1-4q}} = 1 + 2q + 6q^2 + 20q^3 + 70q^4 + 252q^5 + 924q^6 + \cdots = \sum_{n \geq 0} \binom{2n}{n} q^n$$

(18)

$$\frac{1-\sqrt{1-4q}}{2q} = 1 + q + 2q^2 + 5q^3 + 14q^4 + 42q^5 + 132q^6 + \cdots = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} q^n$$

(19)

$$\frac{1}{\sqrt{1-4q}} \left(\frac{1-\sqrt{1-4q}}{2q} \right)^k = \sum_{n \geq 0} \binom{2n+k}{n} q^n$$

(20)

$$\left(\frac{1-\sqrt{1-4q}}{2q} \right)^k = \sum_{n \geq 0} \frac{k(2n+k-1)!}{n!(n+k)!} q^n$$