For binomials, take as definition $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Note that this definition only works then for n an integer, for a more general definition where n may be any real number but k non-negative integer, $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$. Then you may want use the identity that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for the induction steps.

For any problems involving the Fibbonacci numbers, take as definition $F_{-1} = F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

(1)
$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$$

(2) Using the recursive definition of F_n, show that for $n \geq 2$,
$$F_{n+1}^2 = F_{n+1}F_n + F_{n+1}F_{n-1}$$

(3) Show that
$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} .$$

(4) Show that for $n \geq 1$
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1 .$$

(5) Show that for $n > 0$,
$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$$

(6) Show that postage stamps of value 5 cents and 9 cents are sufficient to post any letter requiring more than 31 cents in postage.

(7) If n is any integer such that $n > 4$, then
$$n^2 < 2^n < n! .$$

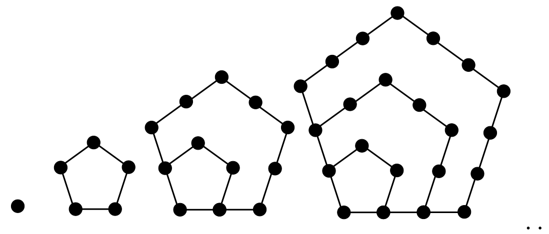
(8) Show that
$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$

(9) Show that for n an integer and θ any real number we have
$$(\cos\ \theta + i \sin\ \theta)^n = \cos\ n\theta + i \sin\ n\theta .$$

(10) Show that for $n, r \geq 0$,
$$\binom{n}{0}\binom{n}{r} + \binom{n}{1}\binom{n}{r+1} + \binom{n}{2}\binom{n}{r+2} + \cdots + \binom{n}{n-r}\binom{n}{n} = \binom{2n}{n+r}$$

(11) Show that
$$1 + 2\binom{n}{1} + 3\binom{n}{2} + \cdots + (k+1)\binom{n}{k} + \cdots + (n+1)\binom{n}{n} = (n+2)2^{n-1}$$

(12) Let $a_n^{(5)}$ be the number of points in n^{th} diagram of the sequence of drawings of nested pentagons shown below. Show that $a_n^{(5)} = \frac{3n^2-n}{2}$.



(13) Show by induction on m, n or r that

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r}$$

(14) Show for $n \geq 0$,

$$\binom{n}{0} F_1 + \binom{n}{1} F_2 + \binom{n}{2} F_3 + \cdots + \binom{n}{n} F_{n+1} = F_{2n+1}$$

(15) Show that for $n \geq 1$,

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} = F_{n+1}$$

The notation $\lfloor x \rfloor$ represents the largest integer less than x.

(16) Show that any positive integer n can be uniquely represented in the form

$$c_1 1! + c_2 2! + c_3 3! + \cdots + c_k k!$$

for some positive integer k where $c_k \neq 0$ and $0 \leq c_i \leq i$ for all $i = 1, 2, \ldots, k$.

(17) Show that any integer n can be uniquely represented in the form

$$r_0 3^0 + r_1 3^1 + r_2 3^2 + \cdots + r_k 3^k$$

for some positive integer k where $r_k \neq 0$ and $r_i \in \{-1, 0, 1\}$ for all $i = 1, 2, \ldots, k$.

(18) Show that $F_n = \frac{1}{\sqrt{5}}(\phi^n - \overline{\phi}^n)$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{1-\sqrt{5}}{2}$.

(19) Show that for all $n \geq 0$,

$$\binom{r-1}{0}\binom{r}{1} + \binom{r-1}{1}\binom{r}{2} + \cdots + \binom{r-1}{r-1}\binom{r}{r} = \binom{4r-1}{2r}$$

(20)

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

(21)

$$F_1 F_2 + F_2 F_3 + F_3 F_4 + \cdots + F_{2n} F_{2n+1} = F_{2n+1}^2 - 1$$

(22) Show that for $n \geq 0$,

$$\left(\frac{1+\sqrt{5}}{2}\right)^n = F_{n+1} + F_n \left(\frac{1-\sqrt{5}}{2}\right)$$

(23) Show that for $n \geq 1$,

$$F_n F_{n+1} = F_{n-1} F_{n+2} + (-1)^{n-1}$$

(24) Show that for $n \geq 0$,

$$F_{n+1}^2 - F_n F_{n+2} = (-1)^{n+1}$$