

## SOME FIBBONACCI/LUCAS GENERATING FUNCTIONS

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The following problems are all connected. Build up the library of generating functions you know by solving for the generating functions in the exercises below.

Recall  $F(q) = \sum_{n \geq 0} F_{n+1}q^n = \frac{1}{1-q-q^2}$  and  $L(q) = \sum_{n \geq 0} L_{n+1}q^n = \frac{1+2q}{1-q-q^2}$

- (1) (a) Use the fact that  $(A(q) + A(-q))/2 = \sum_{n \geq 0} a_{2n}q^{2n}$  to give a generating function for the odd Fibonacci numbers  $F_{odd}(q) = \sum_{n \geq 0} F_{2n+1}q^n$ .
- (b) Use the fact that  $(A(q) - A(-q))/2 = \sum_{n \geq 0} a_{2n+1}q^{2n+1}$  to give a generating function for the even Fibonacci numbers  $F_{even}(q) = \sum_{n \geq 0} F_{2n+2}q^n$ .
- (2) Use the same tricks to find the generating functions for the even and odd Lucas numbers,  $L_{odd}(q) = \sum_{n \geq 0} L_{2n+1}q^n$  and  $L_{even}(q) = \sum_{n \geq 0} L_{2n+2}q^n$ .
- (3) (a) Use the following set of three formulas:

$$\begin{aligned} F_n^2 &= F_n(F_{n-1} + F_{n-2}) = F_n F_{n-1} + F_n F_{n-2} \\ F_n F_{n+1} &= F_n(F_n + F_{n-1}) = F_n^2 + F_n F_{n-1} \\ F_{n+2} F_n &= F_n(F_{n+1} + F_n) = F_{n+1} F_n + F_n^2 \end{aligned}$$

to write down three equations with the generating functions  $D^{(0)}(q) = \sum_{n \geq 0} F_{n+1}^2 q^n$ ,  $D^{(1)}(q) = \sum_{n \geq 0} F_{n+1} F_{n+2} q^n$ ,  $D^{(2)}(q) = \sum_{n \geq 0} F_{n+1} F_{n+3} q^n$ . Use those equations to solve for  $D^{(0)}(q)$ ,  $D^{(1)}(q)$ ,  $D^{(2)}(q)$ .

- (b) Find a formula for  $D^{(3)}(q) = \sum_{n \geq 0} F_{n+1} F_{n+4} q^n$  by replacing  $F_{n+4} = F_{n+3} + F_{n+2}$  and expressing it in terms of  $D^{(2)}(q)$  and  $D^{(1)}(q)$ .
- (4) (a) The Lucas number satisfy the same recurrence as the Fibonacci numbers from the previous problem. Use the same technique to find formulas for  $E^{(0)}(q) = \sum_{n \geq 0} L_{n+1}^2 q^n$ ,  $E^{(1)}(q) = \sum_{n \geq 0} L_{n+1} L_{n+2} q^n$ ,  $E^{(2)}(q) = \sum_{n \geq 0} L_{n+1} L_{n+3} q^n$ .
- (b) Find a formula for  $E^{(3)}(q) = \sum_{n \geq 0} L_{n+1} L_{n+4} q^n$  by replacing  $L_{n+4} = L_{n+3} + L_{n+2}$  and expressing it in terms of  $E^{(2)}(q)$  and  $E^{(1)}(q)$ .
- (5) (a) Use the results of the previous problems and the fact that  $L_n = F_{n+1} + F_{n-1}$  for  $n \geq 2$  to give a formula for the generating function  $M^{(0)}(q) = \sum_{n \geq 0} F_{n+1} L_{n+1} q^n$ .
- (b) Use the generating functions  $D^{(0)}(q)$ ,  $D^{(1)}(q)$ ,  $D^{(2)}(q)$  to give a formula for the generating function  $M^{(1)}(q) = \sum_{n \geq 0} F_{n+2} L_{n+1} q^n$ .
- (c) Use the previous two problems and the fact that  $L_{n+2} = L_{n+1} + L_n$  to find a formula for the generating function  $M^{(-1)}(q) = \sum_{n \geq 0} F_{n+1} L_{n+2} q^n$ .
- (6) (a) Find a formula for  $F_{evensqr}(q) = \sum_{n \geq 0} F_{2n+2}^2 q^n$ .
- (b) Find a formula for  $F_{oddsqr}(q) = \sum_{n \geq 0} F_{2n+1}^2 q^n$ .

Record your answers below:

$$F_{odd}(q) = 1 + 2q + 5q^2 + 13q^3 + 34q^4 + \cdots = \sum_{n \geq 0} F_{2n+1}q^n = \frac{1-q}{1-3q+q^2}$$

$$F_{even}(q) = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n \geq 0} F_{2n+2}q^n = \frac{1}{1-3q+q^2}$$

$$L_{odd}(q) = 1 + 4q + 11q^2 + 29q^3 + 76q^4 + \cdots = \sum_{n \geq 0} L_{2n+1}q^n = \frac{1+q}{1-3q+q^2}$$

$$L_{even}(q) = 3 + 7q + 18q^2 + 47q^3 + 123q^4 + \cdots = \sum_{n \geq 0} L_{2n+2}q^n = \frac{3-2q}{1-3q+q^2}$$

$$D^{(0)}(q) = 1 + q + 4q^2 + 9q^3 + 25q^4 + \cdots = \sum_{n \geq 0} F_{n+1}^2q^n = \frac{1-q}{(1+q)(1-3q+q^2)}$$

$$D^{(1)}(q) = 1 + 2q + 6q^2 + 15q^3 + 40q^4 + \cdots = \sum_{n \geq 0} F_{n+2}F_{n+1}q^n = \frac{1}{(1+q)(1-3q+q^2)}$$

$$D^{(2)}(q) = 2 + 3q + 10q^2 + 24q^3 + 65q^4 + \cdots = \sum_{n \geq 0} F_{n+3}F_{n+1}q^n = \frac{2-q}{(1+q)(1-3q+q^2)}$$

$$D^{(3)}(q) = 3 + 5q + 16q^2 + 39q^3 + 105q^4 + \cdots = \sum_{n \geq 0} F_{n+4}F_{n+1}q^n = \frac{3-q}{(1+q)(1-3q+q^2)}$$

$$E^{(0)}(q) = 1 + 9q + 16q^2 + 49q^3 + 121q^4 + \cdots = \sum_{n \geq 0} L_{n+1}^2q^n = \frac{1+7q-4q^2}{(1+q)(1-3q+q^2)}$$

$$E^{(1)}(q) = 3 + 12q + 28q^2 + 77q^3 + 198q^4 + \cdots = \sum_{n \geq 0} L_{n+2}L_{n+1}q^n = \frac{3+6q-2q^2}{(1+q)(1-3q+q^2)}$$

$$E^{(2)}(q) = 4 + 21q + 44q^2 + 126q^3 + 319q^4 + \cdots = \sum_{n \geq 0} L_{n+3}L_{n+1}q^n = \frac{4+13q-6q^2}{(1+q)(1-3q+q^2)}$$

$$E^{(3)}(q) = 7 + 33q + 72q^2 + 203q^3 + 517q^4 + \cdots = \sum_{n \geq 0} L_{n+4}L_{n+1}q^n = \frac{7+19q-8q^2}{(1+q)(1-3q+q^2)}$$

$$M^{(0)}(q) = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n \geq 0} F_{n+1}L_{n+1}q^n = \frac{1}{1-3q+q^2}$$

$$M^{(1)}(q) = 1 + 6q + 12q^2 + 35q^3 + 88q^4 + \cdots = \sum_{n \geq 0} F_{n+2}L_{n+1}q^n = \frac{1+4q-2q^2}{(1+q)(1-3q+q^2)}$$

$$M^{(-1)}(q) = 3 + 4q + 14q^2 + 33q^3 + 90q^4 + \cdots = \sum_{n \geq 0} F_{n+1}L_{n+2}q^n = \frac{3-2q}{(1+q)(1-3q+q^2)}$$

$$F_{evensqr}(q) = 1 + 9q^2 + 64q^4 + 441q^6 + 3025q^8 + \cdots = \sum_{n \geq 0} F_{2n+2}^2q^n = \frac{1+q}{(1-q)(1-7q+q^2)}$$

$$F_{oddsqr}(q) = 1 + 4q^2 + 25q^4 + 169q^6 + 1156q^8 + \cdots = \sum_{n \geq 0} F_{2n+1}^2q^n = \frac{1-4q+q^2}{(1-q)(1-7q+q^2)}$$

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55$$

$$L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, L_5 = 11, L_6 = 18, L_7 = 29, L_8 = 47, L_9 = 76, L_{10} = 123$$

Using the equations that you found above, find a generating function proof of the following identities for  $n \geq 0$ :

(1)

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n+1}F_{2n+2} = F_{2n+2}^2$$

(2)

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_{n+1}F_{n+2}$$

(3)

$$F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$$

(4)

$$F_2 + F_4 + F_6 + \cdots + F_{2n+2} = F_{2n+3} - 1$$

(5)

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n+2}F_{2n+3} = F_{2n+3}^2 - 1$$

(6)

$$F_{n+1}L_{n+1} = F_{2n+2}$$

(7)

$$F_{n+2}^2 + 2F_{n+1}F_{n+2} = F_{2n+4}$$

(8)

$$F_{n+3}^2 - F_{n+1}^2 = F_{2n+4}$$

(9)

$$F_{n+2}^2 = F_{n+1}F_{n+3} + (-1)^{n+1}$$

(10)

$$F_{n+2}F_{n+3} = F_{n+1}F_{n+4} + (-1)^{n-1}$$

(11)

$$F_{n+2}L_{n+2} + F_{n+1}L_{n+1} = L_{2n+3}$$

(12)

$$F_{n+2}L_{n+2} - F_{n+1}L_{n+1} = F_{2n+3}$$

(13)

$$5(F_{n+1}^2 + F_{n+2}^2) = L_{n+1}^2 + L_{n+2}^2$$

(14)

$$5F_{n+1}^2 - L_{n+1}^2 = 4(-1)^n$$

(15)

$$L_{n+1}^2 - 2L_{2n+2} = -5F_n^2$$

(16)

$$F_{n+4} - F_{n+1} = 2F_{n+2}$$

(17)

$$F_{n+4} + F_{n+1} = 2F_{n+3}$$

(18)

$$F_{n+5} + F_{n+1} = 3F_{n+3}$$

(19)

$$F_{n+5} - F_{n+1} = L_{n+3}$$