COMBINATORIAL PROBLEMS AND GENERATING FUNCTIONS

JANUARY 26, 2012

In each of the following problems the parameter n will be specified. You should write down a generating function for the number of the specified quantity for all n by translating the following combinatorial problems depending on the unknown n into generating functions expressions in a variable q. Use a computer or other means to find the specified coefficient (we will do more exercises soon where we learn to take coefficients without using a computer).

(1) The number of ways are there of distributing $n = 40$ identical jelly beans among four children:
   (a) without restriction
   (b) with all children getting an odd number of jelly beans
   (c) with one child getting at least 10 jelly beans and another child getting at most 10 jelly beans.
   (d) with one child getting an even number of jelly beans and another child getting at most 20 jelly beans
(2) How many ways are there of distributing $n = 10$ loonies amongst 6 people
   (a) without restriction
   (b) such that each person receives at least 1 loonie
   (c) such that the first two people receive the same number of loonies
   (d) such that the first two people receive at most 6 loonies
(3) The number of integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n = 36$ with
   (a) $x_i \geq 0$
   (b) $x_i > 0$
   (c) $x_i \geq i$ (for each $i = 1, 2, 3, 4, 5$)
   (d) with $x_i \geq 0$ and x_1 and x_2 even and x_4 and x_5 odd.
(4) The number of ways to distribute $n = 14$ identical balls into 6 boxes
   (a) with the first two boxes having at least two balls each
   (b) with the first box having *at most* four balls
   (c) with the first two boxes collectively having *at most* four balls.
   (d) with the first three boxes having at most three balls.
(5) How many ways are there of making change for $n = 78$ cents in
   (a) pennies and nickels
   (b) pennies, nickels, and dimes
   (c) pennies, nickels, dimes and quarters
   (d) pennies, nickels, dimes and quarters with at least one of each
(6) The number of election outcomes in the race for class president are there if there are 5 candidates and $n = 75$ students in the class and
   (a) Every candidate receives at least two votes.
   (b) One candidate receives at most one vote and all the other receive at least two votes.
   (c) No candidate receives more than 20 votes.
   (d) Exactly three of the candidates have the same number of votes and they have at least 10 each.

(7) How many ways are there of choosing $n = 10$ coins from an unlimited supply of pennies, nickels, dimes and quarters.
   (a) without restriction
   (b) the number of pennies is equal to the number of nickels
   (c) there are more pennies than there are nickels
   (d) the number of pennies plus the number of nickels is odd

(8) The number of non-negative integer solutions to the equation $x_1+2x_2+3x_3+4x_4 = n = 20$.
   (a) with $x_i \geq 0$
   (b) with $x_i \geq 0$ and $x_1 \leq 6$
   (c) with $4 \geq x_i \geq 0$
   (d) with $x_i \geq i$
   (e) with $4i \geq x_i \geq i$

(9) The number of selections of $n = 50$ marbles from a group of
   (a) 20 reds, 35 blues, and 33 pinks.
   (b) 30 reds, 24 blues and as many greens as you can get.
   (c) 20 reds, any number of blues and greens
   (d) 20 reds, any number of blues and greens but you must select an even number of blues
   (e) 20 reds, as many blues and greens as you need, but the number of blues + the number of greens is even

(10) The number of numbers between 0 and $9,999$ (inclusive) that have
   (a) a sum of digits equal to $n = 16$?
   (b) a sum of digits less than or equal to $n = 16$?
   (c) two even digits and two odd digits and a sum of digits equal to 16?
   (d) More generally, how many numbers between 0 and $10^k - 1$ have a sum of digits equal to n? (give a formula for this number)

(11) The number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n = 20$ with
   (a) $x_i \leq 10$
   (b) $x_1 = 2x_2$
   (c) $x_1 \leq 10$
   (d) $x_1 + x_2 \leq 10$

(12) How many ways are there of distributing $n = 10$ loonies amongst 6 people
   (a) such that the first three people receive 6 loonies in total

(13) The number of integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n = 36$ with
   (a) with $x_i \geq i$ and x_1 and x_2 even and x_4 and x_5 odd.

(14) How many ways are there of making change for $n = 78$ cents in
   (a) Canadian pennies and nickels and U.S. pennies and nickels
   (b) Canadian or U.S. pennies, nickels, dimes and quarters

(15) How many ways are there of rolling 12 different colored (and hence distinguished) six-sided dice so that the sum is $n = 36$.
   (a) no other restriction
   (b) exactly half are odd and half are even
   (c) (challenge) so that exactly 3 different values are showing

(16) How many ways are there of placing $n = 30$ indistinguishable balls in 10 boxes so that
   (a) with no restriction
   (b) the first 4 boxes have at most 10 of the balls.
   (c) first 4 boxes have at least half of the balls

(17) How many combinations of 1, 2, 5, 10 bills make 60 if we are limited to ten 1 bills?