

COMBINATORIAL PROBLEMS AND GENERATING FUNCTIONS

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In each of the following problems the parameter n will be specified. You should write down a generating function for the number of the specified quantity for all n by translating the following combinatorial problems depending on the unknown n into generating functions expressions in a variable q . Use a computer or other means to find the specified coefficient (we will do more exercises soon where we learn to take coefficients without using a computer).

- (1) The number of ways are there of distributing $n = 40$ identical jelly beans among four children:
 - (a) without restriction
 - (b) with all children getting an odd number of jelly beans
 - (c) with one child getting at least 10 jelly beans and another child getting at most 10 jelly beans.
 - (d) with one child getting an even number of jelly beans and another child getting at most 20 jelly beans
- (2) How many ways are there of distributing $n = 10$ loonies amongst 6 people
 - (a) without restriction
 - (b) such that each person receives at least 1 loonie
 - (c) such that the first two people receive the same number of loonies
 - (d) such that the first two people receive at most 6 loonies
- (3) The number of integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n = 36$ with
 - (a) $x_i \geq 0$
 - (b) $x_i > 0$
 - (c) $x_i \geq i$ (for each $i = 1, 2, 3, 4, 5$)
 - (d) with $x_i \geq 0$ and x_1 and x_2 even and x_4 and x_5 odd.
- (4) The number of ways to distribute $n = 14$ identical balls into 6 boxes
 - (a) with the first two boxes having at least two balls each
 - (b) with the first box having *at most* four balls
 - (c) with the first two boxes collectively having *at most* four balls.
 - (d) with the first three boxes having at most three balls.
- (5) How many ways are there of making change for $n = 78$ cents in
 - (a) pennies and nickels
 - (b) pennies, nickels, and dimes
 - (c) pennies, nickels, dimes and quarters
 - (d) pennies, nickels, dimes and quarters with at least one of each
- (6) The number of election outcomes in the race for class president are there if there are 5 candidates and $n = 75$ students in the class and
 - (a) Every candidate receives at least two votes.
 - (b) One candidate receives at most one vote and all the other receive at least two votes.
 - (c) No candidate receives more than 20 votes.
 - (d) Exactly three of the candidates have the same number of votes and they have at least 10 each.

- (7) How many ways are there of choosing $n = 10$ coins from an unlimited supply of pennies, nickels, dimes and quarters.
- (a) without restriction
 - (b) the number of pennies is equal to the number of nickels
 - (c) there are more pennies than there are nickels
 - (d) the number of pennies plus the number of nickels is odd
- (8) The number of non-negative integer solutions to the equation $x_1 + 2x_2 + 3x_3 + 4x_4 = n = 20$.
- (a) with $x_i \geq 0$
 - (b) with $x_i \geq 0$ and $x_1 \leq 6$
 - (c) with $4 \geq x_i \geq 0$
 - (d) with $x_i \geq i$
 - (e) with $4i \geq x_i \geq i$
- (9) The number of selections of $n = 50$ marbles from a group of
- (a) 20 reds, 35 blues, and 33 pinks.
 - (b) 30 reds, 24 blues and as many greens as you can get.
 - (c) 20 reds, any number of blues and greens
 - (d) 20 reds, any number of blues and greens but you must select an even number of blues
 - (e) 20 reds, as many blues and greens as you need, but the number of blues + the number of greens is even
- (10) The number of numbers between 0 and 9,999 (inclusive) that have
- (a) a sum of digits equal to $n = 8$?
 - (b) a sum of digits less than or equal to $n = 8$?
- (11) The number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n = 20$ with
- (a) $x_i \leq 10$
 - (b) $x_1 = 2x_2$
 - (c) $x_1 \leq 10$
 - (d) $x_1 + x_2 \leq 10$
- (12) How many ways are there of distributing $n = 10$ loonies amongst 6 people
- (a) such that the first three people receive 6 loonies in total
- (13) The number of integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n = 36$ with
- (a) with $x_i \geq i$ and x_1 and x_2 even and x_4 and x_5 odd.
- (14) How many ways are there of making change for $n = 78$ cents in
- (a) Canadian pennies and nickels and U.S. pennies and nickels
 - (b) Canadian or U.S. pennies, nickels, dimes and quarters