

SEQUENCES AND SETS OF OBJECTS II

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Say that a_0, a_1, a_2, \dots is a sequence of non-negative integers where a_n represents the number of “widgets of size n .” Assume similarly that b_0, b_1, b_2, \dots is a sequence where b_n represents the number of “doodles of size n .” Each of the following descriptions for a set of elements can be counted using the addition and multiplication principle. For each of the sets below, decompose the description and give an expression in terms the variables a_i and b_i and other arithmetic operations.

- (1) the set of things which are either widgets or doodles of size at most n
- (2) the set of pairs consisting of a widget of size at most n and a doodle which is twice the size of the widget
- (3) the set of pairs consisting of an integer 1 through n and an object of the same size which is either a widget if the integer is even or a doodle if the integer is odd.
- (4) the set of pairs consisting of a widgets of size n and a doodle which is a size strictly smaller than the widget
- (5) the set of pairs consisting of a widget of size at most n and a doodle which is at most half the size of the widget
- (6) the set of pairs consisting of an integer between 1 and n and a widget of the same size as the integer
- (7) the set of pairs consisting of an integer between 1 and n and a widget of size smaller than the integer
- (8) the set of pairs consisting of a widget of size less than or equal to n and an integer between 1 and the size of the widget.
- (9) the set of pairs consisting of a widget of size n and a doodle of size less than or equal to n
- (10) the set of pairs consisting of an integer 1 through n and a widget which is of size in between the integer and n .
- (11) the set of pairs whose first element is a pair consisting of a subset of the integers 1 through n of size 2 and a widget of size less than or equal to n but greater than or equal to the larger of the two numbers in the set.
- (12) the set of triple consisting of two widgets of size n that are different from each other and a doodle of size n
- (13) the set of collections of 3 different widgets of size n (order does not matter)

Write a combinatorial description of a set by combining the sets ‘widgets’ and ‘doodles’ of various sizes such that the number of elements in the set is equal to the following expressions. Your description can be similar to the ones above where we describe “the set of ...” or it can begin with a phrase like “the number of ways of ...”

- (1) $a_1^2 + a_2^2 + \dots + a_n^2$
- (2) $a_1 b_1 + a_2 b_2 + \dots + a_n b_n$
- (3) a_n^2
- (4) a_n^4
- (5) $n^2 a_n$
- (6) $\binom{n}{2} a_n$
- (7) $\binom{2}{2} a_2 + \binom{3}{2} a_3 + \dots + \binom{n}{2} a_n$
- (8) $\binom{n}{0} a_0 + \binom{n}{1} a_1 + \dots + \binom{n}{n} a_n$
- (9) $2a_n b_n$
- (10) $a_k b_{n-k}$
- (11) $a_1 + a_3 + a_5 + \dots + a_{2n-1}$
- (12) $a_0 + a_2 + a_4 + \dots + a_{2n}$
- (13) $a_n^{b_n}$
- (14) $a_1^{b_1} + a_2^{b_2} + \dots + a_n^{b_n}$
- (15) $a_n!$