UNEXAM #4

ASSIGNED: MARCH 23, 2008, DUE: APRIL 10, 2008

Remember that the important aspect of this assignment is not the answer, but the solution. Please justify the following.

I will be grading you on a 5 point scale again. Please provide me with a single, clear, short solution which includes all details. Each of these problems boils down to essentially one thing: organize the question in a way that you can apply the multiplication or addition principle of generating functions. When you explain this you should tell me
(1) how you are organizing 'what you want' in a way you can apply basic principles of generating functions
(2) how to write down the generating function for each piece of the expressions
(3) how to put these generating functions together using the addition and multiplication principle.
Do not bother to try to explain algebra to me (in fact, I would prefer if you leave off any serious algebra from your explanation). That is not the point of this exercise.

When you finish your explanations. Please go back and reread and verify that your answer satisfies the following criteria:
(1) is it clear what question you are answering?
(2) are you assuming basic facts from somewhere else, if so tell me from where?
(3) Is your explanation complete from the perspective of someone that does not know what the assignment is?

(1) Show that for $n \geq 1$

$$F_n^2 + F_{n+1}^2 = F_{2n+1} .$$

Answer: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$.
$F_1^2 + F_2^2 = 1^2 + 1^2 = 2 = F_3$
$F_2^2 + F_3^2 = 1^2 + 2^2 = 5 = F_5$
$F_3^2 + F_4^2 = 2^2 + 3^2 = 13 = F_7$
$F_4^2 + F_5^2 = 3^2 + 5^2 = 34 = F_9$
and probably this pattern continues...

(2) Explain why the following identity is true for $n \geq 0$,

$$F_{2n+2} = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} 3^{n-2i}$$

Answer: $F_2 = 1 = \binom{0}{0}3^0$
$F_4 = 3 = \binom{1}{0}3^1$
$F_6 = 8 = \binom{2}{0}3^2 - \binom{1}{1}3^0$
$F_8 = 21 = \binom{3}{0}3^3 - \binom{2}{1}3^1$
$F_{10} = 55 = \binom{4}{0}3^4 - \binom{3}{1}3^2 + \binom{2}{2}3^0$
and probably this pattern continues...

(3) Explain why the following identity is true

$$1 + \sum_{k \geq 1} \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{i \geq 1} \frac{1}{1-q^i}$$

Answer: set $z = q$ in the identity (13-2-2) on page 167 of G. Andrews *Number Theory*.