MATH 6121 lecture notes

Transcribed and formatted by John M. Campbell
jmaxwellcampbell@gmail.com

2 September 13 lecture

2.1 Project information

Pick a theorem based on the subject matter of MATH 6121, and write a corresponding computer program (e.g. using SageMath, Maple, Mathematica, MATLAB, C++, etc.).

1. Decide what theorem/construction you want to implement, and come up with a proposal.
2. Write functions/programs to compute the theorem/construction. It needs to be well-documented. You need to provide:
   - A description of each function used;
   - A description of the input variables and the output; and
   - Examples of how the functions work.
3. Give a simple example of your program in use.

Here is an illustration of an idea for a subject for this project. Write a function which computes the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ as output, where $\mathcal{B}$, $\mathcal{C}$, and T are given by the following input:

(i) An input basis $\mathcal{B}$ for a vector space V;

(ii) An output basis $\mathcal{C}$ for a vector space W; and

(iii) A linear transformation $T: V \to W$.

In particular, consider a SageMath implementation for this function, e.g. with a SageMath function

```
lin_trans_to_matrix(B, C, T)
```

the output of which is a matrix.

2.2 Direct sums of vector spaces

V finitely generated $\Longleftrightarrow \mathbb{C}^n$.

Given a basis $\mathcal{B}$, with $L_{\mathcal{B}}\colon V \to \mathbb{C}^n$, and $T\colon V \to W$, and a basis $\mathcal{C}$ for W, then ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ is a matrix of dimension $\dim(W) \times \dim(V)$.

If $M = {}_{\mathcal{C}}[T]_{\mathcal{B}}$ then $L_{\mathcal{C}} \circ T = M \circ L_{\mathcal{B}}$.

Exercise 2.1. If $\vec{u} \in \mathbb{C}^n$ and $M\vec{u} = \vec{0}_{\mathbb{C}^m}$, then show that $T(L_{\mathcal{B}}^{-1}(\vec{u})) = \vec{0}_W$.

Direct sum: $V \oplus W = \{(v, w) : v \in V, w \in W\}$.

If $\mathcal{B}_V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $\mathcal{B}_W = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ are bases for V and W, then

$$\mathcal{B}_{V\oplus W} = \{(\vec{v}_1, 0), (\vec{v}_2, 0), \ldots, (\vec{v}_n, 0), (0, \vec{w}_1), (0, \vec{w}_2), \ldots, (0, \vec{w}_m)\}$$

is a basis for $V \oplus W$, with addition and scalar multiplication defined as follows.

$$\begin{array}{lr} +_{\oplus} & (\vec{v}, \vec{w}) +_{\oplus} (\vec{x}, \vec{y}) = (\vec{v} + \vec{x}, \vec{w} + \vec{y}) \\ \cdot_{\oplus} & c \cdot_{\oplus} (\vec{v}, \vec{w}) = (c\vec{v}, c\vec{w}). \end{array}$$

Exercise 2.2. Check that $V \oplus W$ is a vector space with respect to the above operations.

Take $T\colon V \to X$ and $Q\colon W \to Y$.

Then define

$$T \oplus Q\colon V \oplus W \to X \oplus Y$$

so that

$$(\vec{v}, \vec{w}) \mapsto (T(\vec{v}), Q(\vec{w}))$$

for $\vec{v} \in V$ and $\vec{w} \in W$.

Problem 2.3. What is ${}_{\mathcal{B}_{X\oplus Y}}[T \oplus Q]_{\mathcal{B}_{V\oplus W}}$?

Exercise 2.4. Let $\dim(V) = n$, $\dim(W) = m$, $\dim(X) = r$, and $\dim(Y) = s$. Prove that ${}_{\mathcal{B}_{X\oplus Y}}[T \oplus Q]_{\mathcal{B}_{V\oplus W}}$ is equal to the following $(r+s) \times (n+m)$ matrix.

$$\begin{array}{cc} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} r \\ s \end{array} & \left[\begin{array}{c|c} {}_{\mathcal{B}_X}[T]_{\mathcal{B}_V} & 0 \\ \hline 0 & {}_{\mathcal{B}_Y}[Q]_{\mathcal{B}_W} \end{array}\right] \end{array}$$

Recall that the matrix ${}_{\mathcal{B}_X}[T]_{\mathcal{B}_V}$ may be defined as

$$ {}_{\mathcal{B}_X}[T]_{\mathcal{B}_V} = \left[L_{\mathcal{B}_X}(T(\vec{v}_1)), L_{\mathcal{B}_X}(T(\vec{v}_2)), \ldots, L_{\mathcal{B}_X}(T(\vec{v}_n))\right], $$

where $T\colon V \to W$ denotes a linear transformation, and $L_{\mathcal{B}_X}: W \to \mathbb{C}^r$ is as defined in the previous lecture.

2.3 Tensor products of vector spaces

Definition 2.5. Let V and W be vector spaces, and let $\mathcal{B}_V$ and $\mathcal{B}_W$ respectively denote bases of V and W. Then we define the tensor product $V \otimes W$ of V and W as

$$ \operatorname{span}\{\vec{v}_i \otimes \vec{w}_j : \vec{v}_i \in \mathcal{B}_V, \vec{w}_j \in \mathcal{B}_W\} $$

so that

$$ (a\vec{v} + b\vec{x}) \otimes \vec{w} = a(\vec{v} \otimes \vec{w}) + b(\vec{x} \otimes \vec{w}) $$

and

$$ \vec{v} \otimes (a\vec{w} + b\vec{x}) = a(\vec{v} \otimes \vec{w}) + b(\vec{v} \otimes \vec{x}). $$

Remark 2.6. Intuitively, you can think of the tensor product $V \otimes W$ as consisting of "*pairs with special properties*". Intuitively, you can think of taking tensor products as "*a way of grouping things together*". Taking the direct sum of vector spaces and taking the tensor product of vector spaces both involve the important concept of "*building larger vector spaces from smaller ones*". In order to intuitively understand that the direct sum of vector spaces and the tensor product of vector spaces are *very different constructions*, you need to "*distinguish how the pairs are created*".

Proposition 2.7. *Let V be a vector space of dimension n and let W be a vector space of dimension m. Then*

$$ V \oplus W = (\dim(V)) + (\dim(W)) = n + m $$

and

$$ V \otimes W = (\dim(V))(\dim(W)) = nm. $$

Exercise 2.8. Let $V = \mathbb{R}^2$, and let $W = \mathbb{R}^2$. With respect to the tensor product $V \otimes W$, show that:

$$ (1,1) \otimes (1,4) + (1,-2) \otimes (-1,2) = 0\,(1,0) \otimes (1,0) + $$

$$6\,(1,0)\otimes(0,1)+$$
$$3\,(0,1)\otimes(1,0)+$$
$$0\,(0,1)\otimes(0,1)\,.$$

With respect to the direct sum $V\oplus W$, show that

$$((1,1)\,,(1,4))+((1,-2)\,,(-1,2))=((2,-1)\,,(0,6))\,.$$

Question 2.9. What does it mean to take a linear transformation on a tensor?

Let $T\colon V\to X$ and $Q\colon W\to Y$ be linear transformations given as follows:

$$T(\vec{v}_i)=\sum_{j=1}^{r}a_{i,j}\vec{x}_j$$
$$Q(\vec{w}_k)=\sum_{\ell=1}^{s}b_{k,\ell}\vec{y}_\ell.$$

We thus define the mapping

$$T\otimes Q\colon V\otimes W\to X\otimes Y$$

as follows:

$$(T\otimes Q)\,(\vec{v}_i\otimes\vec{w}_k)=\sum_{j=1}^{r}\sum_{\ell=1}^{s}a_{i,j}b_{k,\ell}\,(\vec{x}_j\otimes\vec{y}_\ell)\,.$$

The matrix corresponding to this linear transformation may be defined using the **Kronecker product**.

Basis for $V\otimes W$:

$$\mathcal{B}_{V\otimes W}=\{\vec{v}_1\otimes\vec{w}_1,\vec{v}_1\otimes\vec{w}_2,\ldots,\vec{v}_1\otimes\vec{w}_m,\vec{v}_2\otimes\vec{w}_1,\ldots,\vec{v}_n\otimes\vec{w}_m\}\,.$$

If ${}_{\mathcal{B}_X}[T]_{\mathcal{B}_V}=A$ and ${}_{\mathcal{B}_Y}[Q]_{\mathcal{B}_W}=B$, then the matrix

$${}_{\mathcal{B}_{X\otimes Y}}[T\otimes Q]_{\mathcal{B}_{V\otimes W}}$$

is equal to the following Kronecker product of matrices.

$$\begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1}B & a_{r,2}B & \cdots & a_{r,n}B \end{bmatrix}$$

Exercise 2.10. Let $\mathcal{B}_V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{B}_W = \{\vec{w}_1, \vec{w}_2\}$. Let $\phi\colon V \to V$ be such that

$$\phi\left(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3\right) = c\vec{v}_1 + 2a\vec{v}_2 - 3b\vec{v}_3,$$

and let $\psi\colon W \to W$ be such that

$$\psi\left(a\vec{w}_1 + b\vec{w}_2\right) = (a + 3b)\,\vec{w}_1 + (4b - 2a)\,\vec{w}_2.$$

Compute ${}_{\mathcal{B}_V}[\phi]_{\mathcal{B}_V}$, ${}_{\mathcal{B}_W}[\psi]_{\mathcal{B}_W}$, and

$${}_{\mathcal{B}_{V\otimes W}}[\phi \otimes \psi]_{\mathcal{B}_{V\otimes W}}.$$

Note that $\mathcal{B}_{V\otimes W}$ consists of six elements that have a specific order.

2.4 Basic group theory

Definition 2.11. A **group** is a set G endowed with an associative binary operation $\circ_G$ such that there is an element $e \in G$ such that

$$e \circ_G a = a \circ_G e = a$$

and for every $a \in G$, there exists an element $a^{-1} \in G$ such that $a \circ_G a^{-1} = e$.

Example 2.12. $(\mathbb{Z}, +)$ is a group.

Example 2.13. $(\mathbb{Q}, +)$ is a group.

Example 2.14. $(\mathbb{R}, +)$ is a group.

Example 2.15. $(\mathbb{C}, +)$ is a group.

Example 2.16. $(\mathbb{R} \setminus \{0\}, \cdot)$ is a group.

Remark 2.17. $(\mathbb{Z} \setminus \{0\}, \cdot)$ is not a group.

Definition 2.18. The **cyclic group** C_n is the group with an underlying set of the form $\{1, a, a^2, \ldots, a^{n-1}\}$ such that $a^r \cdot a^s = a^{r+s}$ and $a^n = 1$.

Definition 2.19. The **dihedral group** D_n is the group with an underlying set of the form

$$\left\{1, a, a^2, \ldots, a^{n-1}, b, ba, ba^2, \ldots, ba^{n-1}\right\}$$

such that $\{1, a, a^2, \ldots, a^{n-1}\}$ is cyclic, and $ba = a^{-1}b$, and $b^2 = 1$.

The **symmetric group** S_n (*or* more generally S_X, where X is a finite set) consists of permutations on an n-element set, such that the compositions of permutations is the underlying binary operation:

$$S_X = (\text{permutations of } X, \circ),$$

where $\circ$ denotes the composition operation with respect to permutations.

Recall that the composition of permutations $\sigma\colon X \to X$ and $\tau\colon X \to X$ in S_X is given by composing τ and σ as functions. For example, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 1 & 6 & 7 & 2 & 4 \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix},$$

and consider the product $\tau \circ \sigma \in S_7$. To evaluate the composition $\tau \circ \sigma$, begin by evaluating the expression $(\tau \circ \sigma)(1)$:

$$(\tau \circ \sigma)(1) = \tau(\sigma(1)) = \tau(3) = 4.$$

Similarly, we have that:

$$(\tau \circ \sigma)(2) = \tau(\sigma(2)) = \tau(5) = 6.$$

Continuing in this manner, we have that:

$$\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 2 & 7 & 1 & 3 & 5 \end{pmatrix}.$$

Definition 2.20. A **homomorphism** on groups $G \to H$ is a map $\phi\colon G \to H$ such that $\phi(g_1 \circ_G g_2) = \phi(g_1) \circ_H \phi(g_2)$.

Definition 2.21. A homomorphism which is a bijection is an **isomorphism**.

Problem 2.22. Given a natural number n, how many groups of order n are there up to isomorphism?

# of elements in G	1	2	3	4	5	6	7
# of groups up to isomorphism	1	1	1	2	1	2	1

The above integer sequence is described in the **On-Line Encyclopedia of Integer Sequences** (OEIS). In particular, this sequence given by the OEIS sequence labeled A000001 (see `http://oeis.org/A000001`).

Letting $a_n = \text{A000001}_n$ denote the number of groups of order $n \in \mathbb{N}$ up to isomorphism, the value of a_n is given below for $n = 1, 2, \ldots, 17$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a_n	1	1	1	2	1	2	1	5	2	2	1	5	1	2	1	14	1

Next lecture: groups acting on a set, groups acting on a vector space.