MATH 6121 lecture notes

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4.1 Errata

There was an error in the previous formulation of Exercise 9, which has been corrected in the online lecture notes. For this exercise, you need to prove that the mapping

$$\phi_x \colon \text{Orbit} \to G/\text{Stab}(x)$$

whereby $g.x \mapsto g\text{Stab}(x)$ is a well-defined G-set homomorphism and an isomorphism. This exercise initially required showing that this mapping is a group homomorphism, instead of a G-set homomorphism.

To prove that this mapping is a G-set homomorphism, you need to show that $g\phi_x(g'.x) = \phi((gg').x)$.

However, it doesn't make sense to consider $\phi_x \colon \text{Orbit}(x) \to G/\text{Stab}(x)$ as a group homomorphism, since $G/\text{Stab}(x)$ does not necessarily have the structure of a group in the sense that $\text{Stab}(x)$ is not necessarily a normal subgroup of G.

There was also an error in the definition of the term **group action** given in the previous lecture which has been corrected in the online lecture notes.

A G-set is defined as a set X with an operation . from $G \times X$ to X such that $g_1.(g_2.x) = (g_1g_2).x$ and $e.x = x$. This second axiom wasn't included in the definition given in class.

It is important to include this second axiom in the definition of a group action.

To illustrate why it is important to include the axiom given by the equality $e.x = x$ in the definition of a group action, let $X = \{x, y\}$, and let G denote a finite group such that $|G| > 1$. Define the mapping . so that $g.x = y$ and $g.y = y$. This mapping turns out to be very different compared to group actions on G, and the equality $g.x = y$ intuitively makes it difficult to "work with" the mapping . when dealing with homomorphisms onto the symmetric group.

4.2 MATH 6121 projects

Present your project in a way that "shows off" the theorem your project is based upon.

Project proposal: explicitly state which theorem you are interested in implementing.

Write the functions themselves used in your computer program in your documentation.

There should be lots of documentation corresponding to your computer program.

Explain where you use the theorem you chose for your project. Explain applications of this theorem, and applications of your program.

You should consider using a variety of "**helper functions**", i.e. "secondary" functions used with respect to your main computer program. Write out documentation and explanations for helper functions.

Explain how to use your program through examples, illustrations, etc.

It is important to *incude computer output in your examples.*

4.3 Illustration of a project idea

The following questions motivate a possible topic proposal.

Question 4.1. How can a vector be represented, e.g. through the use of different data structures?

Question 4.2. What does an abstract vector space intuitively "look like"?

Question 4.3. What does a linear transformation "look like" when you turn it into a matrix?

The example of SageMath code on the course webpage shows how to use SageMath to take a linear transformation which is listed as a function and turn it into a matrix.

The formula used in this code is precisely the formula given in class for computing matrices for linear transformations.

4.3.1 Representing vectors using dictionary data structures

A vector may be represented as a **dictionary** in SageMath.

A dictionary is a certain type of data structure used in SageMath.

According to Wikipedia: "In computer science, an **associative array**, **map**, **symbol table**, or **dictionary** is an abstract data type composed of a collection of (key, value) pairs, such that each possible key appears at most once in the collection."

Informally, a really good way of "holding" a vector would be through the use of a dictionary data type.

Other data structures may also be used to represent vectors through the use of computer programs. For example, certain data structures based on lists or column vectors may be used to represent the elements in an abstract vector space.

Your choice of data structures to represent abstract mathematical objects depends in part on your choice of programming languages.

It is not a good idea to use "low-level" programming languages such as C or Fortran when dealing with algebraic structures such as groups, vector spaces, rings, etc.

In contrast, computer algebra systems such as SageMath already have many different types of algebraic objects "built in" to the corresponding language.

4.4 Burnside's lemma

Question 4.4. How many ways are there of coloring the 8×8 grid with black and white such that elements in the same orbit are counted once?

Again let the dihedral group D_4 be denoted as follows:

$$D_4 = \left\{1, a, a^2, a^3, b, ba, ba^2, ba^3\right\}.$$

Recall that Burnside's lemma or Burnside's formula may be formulated in the following manner:

$$\begin{aligned}\#\text{ of orbits} &= \frac{1}{|G|}\sum_{g\in G}(\#\text{ of colorings fixed by } g)\\ &= \frac{1}{|G|}\sum_{x\in\text{colorings}}|\mathrm{Stab}(x)|\,.\end{aligned}$$

Observe that there are 2^{64} colorings fixed by 1.

Recall that a denotes the isometry given by rotation by 90° clockwise.

How may colorings are fixed by $a \in D_4$?

A coloring fixed by $a \in D_4$ would have to be of the following form.

1	2	3	4	13	9	5	1
5	6	7	8	14	10	6	2
9	10	11	12	15	11	7	3
13	14	15	16	16	12	8	4
4	8	12	16	16	15	14	13
3	7	11	15	12	11	10	9
2	6	10	14	8	7	6	5
1	5	9	13	4	3	2	1

What type of isometry is given by the composition $b \circ_{D_4} a = ba \in D_4$?

The isometry $ba \in D_4$ is given by rotation by 90° clockwise followed by a vertical "flip" or reflection.

It is easily seen that the isometry given by the product $ba \in D_4$ is given by a diagonal reflection, i.e. a "flipping" across the "main diagonal" of the Euclidean plane $\mathbb{E}$.

So, how many colorings are fixed by $ba \in D_4$?

A coloring fixed by $ba \in D_4$ would have to be of the following form.

1	2	3	4	5	6	7	8
2	9	10	11	12	13	14	15
3	10	16	17	18	19	20	21
4	11	17	22	23	24	25	26
5	12	18	23	27	28	29	30
6	13	19	24	28	31	32	33
7	14	20	25	29	32	34	35
8	15	21	26	30	33	35	36

We thus obtain the following data which may be used to answer Question 4.4.

Element in D_4	# of fixed colorings
1	2^{64}
a	$2^{\frac{64}{4}}$
a^2	$2^{\frac{64}{2}}$
a^3	same as a
b	same as a^2
ba	2^{36}
ba^2	$2^{\frac{64}{2}}$
ba^3	2^{36}

So, in answer to Question 4.4, by Burnside's lemma, the number of colorings of an 8×8 grid with black and white such that elements in the same orbit are counted once is equal to:

$$\frac{1}{8}\left(2^{64} + 2^{16} + 2^{32} + 2^{16} + 2^{32} + 2^{36} + 2^{32} + 2^{36}\right)$$
$$= 2305843028004192256.$$

4.4.1 Generalizations

The OEIS sequence A054247 (see `https://oeis.org/A054247`) is the integer sequence given by the number of $n \times n$ binary matrices under the action of the dihedral group D_4. Letting A054247_n denote the n^{th} entry in this sequence for $n \in \mathbb{N}$, from our above results, we thus have that:

$$\text{A054247}_8 = 2305843028004192256.$$

As indicated in the OEIS sequence A054247, there is an elegant closed-form formula, given below, for the number A054247_n of $n \times n$ binary matrices under the action of the dihedral group D_4. This formula is easily verified using Burnside's lemma.

$$\text{A054247}_n = \begin{cases} \frac{1}{8}\left(2^{n^2} + 2^{\frac{n^2}{4}+1} + 3 \times 2^{\frac{n^2}{2}} + 2^{\frac{n^2+n}{2}+1}\right) & \text{if } n \text{ is even} \\ \frac{1}{8}\left(2^{n^2} + 2^{\frac{n^2+3}{4}+1} + 2^{\frac{n^2+1}{2}} + 2^{\frac{n^2+n}{2}+2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Also observe that the number of different patterns given by colourings of an 8×8 array with c different colors is

$$\frac{1}{8}\left(c^{64} + 2c^{36} + 3c^{32} + 2c^{16}\right)$$

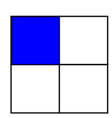
as may be verified using Burnside's lemma[1].

4.5 Normal subgroups

We presently return the the problem described in the previous lecture given by enumerating different patterns given by colorings of 2×2 grids with two colors.

Recall that we are letting the dihedral group D_4 be given as follows: $D_4 = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$.

We computed the following stabilizer-subgroup in the previous lecture:

Stab () $= \{1, ba\}$.

Now compute the quotient group given by the above subgroup of D_4 as indicated below.

[1]See Slomson's *Introduction to Combinatorics*, p. 162.

$$D_4 \Big/ \mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) = \left\{\{1, ba\}, \{a, b\}, \left\{a^2, ba^3\right\}, \left\{a^3, ba^2\right\}\right\}.$$

Evaluate each element in the above set as follows.

$$\begin{aligned}
\{1, ba\} &= \mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) = ba \cdot \mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) \\
\{a, b\} &= a \cdot \mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) = b \cdot \mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) \\
\left\{a^2, ba^3\right\} &= a^2\mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) = ba^3\mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) \\
\left\{a^3, ba^2\right\} &= a^3\mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right) = ba^2\mathrm{Stab}\left(\begin{array}{|c|c|}\hline \blacksquare & \\ \hline & \\ \hline \end{array}\right)
\end{aligned}$$

The group D_4 acts on the above stabilizer subgroup by left-multiplication.

For example, simplify the expression ba^2ba using the dihedral relations $ab = ba^3$, $b^2 = 1$, and $a^4 = 1$.

$$\begin{aligned}
ba^2ba &= ba(ab)a \\
&= ba(ba^3)a \\
&= baba^4 \\
&= b(ab) \\
&= b(ba^3) \\
&= b^2a^3 \\
&= a^3.
\end{aligned}$$

The above set of cosets is a D_4-set, but it does not have the structure of a group. For example, since

$$\{1, ba\} \cdot \{a, b\} = \{1 \cdot a, 1 \cdot b, baa, bab\} = \left\{a, b, ba^2, a^3\right\}$$

is of order 4, we have that "element-wise" multiplication is not a binary operation on the above set consisting of order-2 cosets.

Now let $H = \{1, a^2\}$, and observe that $H \leq D_4$. Compute the quotient D_4/H:

$$D_4/H = \left\{\left\{1, a^2\right\}, \left\{a, a^3\right\}, \left\{b, ba^2\right\}, \left\{ba, ba^3\right\}\right\}.$$

Now, if we multiply these cosets together in the "element-wise" manner illustrated above, we actually *do* obtain a group structure, as illustrated in the following Cayley table.

This is precisely because $H \trianglelefteq D_4$.

$\circ_{D_4/H}$	$\{1, a^2\}$	$\{a, a^3\}$	$\{b, ba^2\}$	$\{ba, ba^3\}$
$\{1, a^2\}$	$\{1, a^2\}$	$\{a, a^3\}$	$\{b, ba^2\}$	$\{ba, ba^3\}$
$\{a, a^3\}$	$\{a, a^3\}$	$\{1, a^2\}$	$\{ba, ba^3\}$	$\{b, ba^2\}$
$\{b, ba^2\}$	$\{b, ba^2\}$	$\{ba, ba^3\}$	$\{1, a^2\}$	$\{a, a^3\}$
$\{ba, ba^3\}$	$\{ba, ba^3\}$	$\{b, ba^2\}$	$\{a, a^3\}$	$\{1, a^2\}$

There is a natural correspondence between the above Cayley table and the group illustrated in the following Cayley table. This group is isomorphic to the dihedral group D_2, which is isomorphic to the Klein four-group $C_2 \times C_2$. This group may be defined by the following relations: $r^2 = 1$, $s^2 = 1$, and $rs = sr$.

$\cdot$	1	r	s	rs
1	1	r	s	rs
r	r	1	rs	s
s	s	rs	1	r
rs	rs	s	r	1

There is an obvious isomorphism between the two groups illustrated above given as follows:

$$\left\{1, a^2\right\} \mapsto 1$$
$$\left\{a, a^3\right\} \mapsto r$$
$$\left\{b, ba^2\right\} \mapsto s$$
$$\left\{ba, ba^3\right\} \mapsto rs.$$

Recall that G/H denotes the set of left cosets of H, with $G/H = \{gH : g \in G\}$, where $gH = \{gh : h \in H\}$.

Multiplication between cosets may be defined so that the multiplication is between all elements in the sets.

Exercise 10: Prove that if $H \trianglelefteq G$, then G/H forms a group with respect to the operation $\circ_{G/H}$ on G/H whereby $g_1H \circ_{G/H} g_2H = g_1g_2H$ for all $g_1, g_2 \in G$.

What exactly do you need to do to show that G/H forms a group with respect to the operation $\circ_{G/H}$ given above?

You need to show that this operation is *well-defined* in the sense that this operation does not depend on the coset representatives of its input.

There are many ways of writing down a coset. Observe that $g_1H = g_1hH$ for arbitrary $h \in H$.

Since there are many different ways of "writing down a coset", you need to show that $\circ_{G/H}$ does not depend on specific coset representatives.

Using the fact that $H \trianglelefteq G$, it can be shown that the following equality holds.

$$g_1Hg_2H = \{g_1h_2g_2h_2 : h_1, h_2 \in G\} = g_1g_2H.$$

Also recall that a subgroup $H \leq G$ is a normal subgroup if and only if $gH = Hg$ for all $g \in G$.

Also observe that if G/H forms a group under an appropriately defined binary operation, then H must be normal in G.

Exercise 11: Show that $\phi\colon G \to G/H$ is a group homomorphism, where $g \mapsto gH$, and $\ker(\phi) = H$.

Remark 4.5. The surjective homomorphism $\phi\colon G \to G/H$ given in the above exercise is often referred to as the **canonical morphism** or **canonical projection** from G to G/H.

It is important to note that *normal subgroups are precisely kernels of group homomorphisms*. In other words, $H \trianglelefteq G$ *if and only if* H is the kernel of a group homomorphism.