DETAIL OF p-GROUP PROOF

MIKE ZABROCKI

On Sept 29, I was trying to prove the theorem below and there was one detail that I was missing. I wrote 'exercise' on that point and moved on. Here is the proof with that detail filled in.

Theorem 1. *Given a p-group G and an $H \lneq G$, then $H \neq N_G(H)$.*

Proof. We had to choose a $K \trianglelefteq G$ and maximal such that $K \leq H$ and G/K is non-trivial (and we know there is at least one because $K = \{1\}$ has this property). If $H \trianglelefteq G$ (that is, $K = H$), then (the detail that I was missing in class was that) $N_G(H) = G \neq H$ and this shows the theorem.

Now consider subgroup Z' of G corresponding to $Z(G/K)$ by the 4^{th} isomorphism theorem. So pick an element $z \in Z'$ (or $zK \in Z(G/K)$) and since $hzK = hK \cdot_{G/K} zK = zK \cdot_{G/K} hK = zhK$, so $z^{-1}hz \in hK \subset hH = H$, hence $z \in N_G(H)$ and $Z' \subseteq N_G(H)$. Either $Z' \neq H$ and $N_G(H) \neq H$, or $Z' = H$. In the latter case, a similar argument[1] shows that $H \trianglelefteq G$. $\square$

[1] $\forall g \in G$, $hgK = hK \cdot_{G/K} gK = gK \cdot_{G/K} hK = ghK$ and $g^{-1}hg \in hK \subset hH = H$ and $H \trianglelefteq G$