# GROUP THEORY HOMEWORK

MATH 6121

(1) Recall that for groups $A$ and $B$ and $\gamma : B \to Aut(A)$, then the group $A \rtimes_\gamma B$ is the set of pairs $\{(a,b) : a \in A, b \in B\}$ with product $(a,b) \cdot_{A \rtimes_\gamma B} (a',b') = (a\gamma_b(a'), bb')$. Find an example of $p, q$ and $\gamma$ such that $\mathbb{Z}_p \rtimes_\gamma \mathbb{Z}_q$ is solvable but not abelian.

(2) Consider the following series:

$$Z_0(G) := \{1\}$$

$$Z_1(G) = Z(G) = \{g \in G | gx = xg, \forall g \in G\} \trianglelefteq G$$

Then, we recall the map $\pi : G \to G/Z_1(G)$, and note that $Z(G/Z1(G)) \trianglelefteq G/Z_1(G)$. Thus, we may use the fourth isomorphism theorem to create $Z_2(G)$ as the group corresponding to $Z(G/Z1(G))$, and we have that $Z_1(G) \trianglelefteq Z_2(G)$, in general we have $Z_i(G)$ as the subgroup corresponding to $Z(G/Z_{i-1}(G))$, and $Z_{i-1}(G) \trianglelefteq Z_i(G) \trianglelefteq G$. Hence we have

$$Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \cdots$$

which is called the **Upper Central Series** of $G$. If $G$ is a nontrivial finite group then this series stabilizes at some point.

**Def:** We say that $G$ is **Nilpotent** of index $k$, if there exists a $k$ such that $Z_k(G) = G$ and $Z_{k-1}(G) \neq G$.

So, for $G$ a finite group, show that all abelian groups are nilpotent, and that all nilpotent groups are solvable. We note that the converse of each statement is not true.

(3) We assume that G is a finite nontrivial group. And define $G^0 = G$ and

$$G^1 = [G,G] := \left\langle ghg^{-1}h^{-1} | g, h \in G \right\rangle \ .$$

Show that $[G,G] \trianglelefteq G$ and $G/[G,G]$ is abelian. Furthermore, show that if $H \triangleleft G$ such that $G/H$ is abelian then $[G,G] \leq H$. Let

$$G^i = [G, G^{i-1}] = \left\langle ghg^{-1}h^{-1} | g \in G, h \in G^{i-1} \right\rangle \ .$$

Show that $G^i \trianglelefteq G$, then we have the Lower Central Series of G given by

$$\ldots \trianglelefteq G^2 \trianglelefteq G^1 \trianglelefteq G \ ,$$

which will stabilize at some point since $G$ is finite. Show that $G^k = \{1\}$ if and only if G is nilpotent of index $k$. (Hint: $Z_i(G) \leq G^{k-i-1} \leq Z_{i+1}(G)$.)

(4) Let $G$ be a nontrivial finite group. Then define $G^{(0)} = G$ and

$$G^{(i)} = [G^{(i-1)}, G^{(i-1)}] \trianglelefteq G^{(i-1)}.$$

We note that it is not true in general that $G^{(i)} \trianglelefteq G$. We call the following normal series a **Derived Series**, and since $G$ is finite it stabilizes at some point:

$$G^{(k)} \trianglelefteq \cdots \trianglelefteq G^{(2)} \trianglelefteq G^{(1)} \trianglelefteq G^{(0)}.$$

Show that $G$ is solvable if and only if $G^{(k)} = \{1\}$ for some $k$.
(Hint: Use the fact that $G^{(i+1)}/G^{(i)}$ is abelian, this implies that $G^{(i+1)} = [H_{s-i}, H_{s-i}] \trianglelefteq H_{s-i-1}$ where $\{1\} \trianglelefteq H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_k = G$).
What can you say about groups of order...

(5) ... 2014? Are they simple? abelian? nilpotent? solvable?
(6) ... 2015? Are they simple? abelian? nilpotent? solvable?
(7) ... 2016? Are they simple? abelian? nilpotent? solvable?

Note: $2014 = 2 \cdot 19 \cdot 53$, $2015 = 5 \cdot 13 \cdot 31$, $2016 = 2^5 \cdot 3^2 \cdot 7$.