MATH 6121 lecture notes

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Cauchy's theorem may be used to simplify the discussion given towards the end of the September 29th lecture. Recall that Cauchy's theorem may be formulated in the following manner (see Fraleigh's "A First Course in Abstract Algebra"):

Cauchy's theorem: Let p be a prime. Let G be a finite group and let p divide $|G|$. Then G has an element of order p and, consequently, a subgroup of order p.

12.1 Elementary character theory

Characters are the essence of representations.

Once you know the behaviour of a character function, you know everything you need to know about the representation.

Let G denote the multiplicative cyclic group $\{e, a, a^2\}$. Using Maschke's theorem, we have shown that:

$$\begin{aligned} V &= \mathscr{L}\{e_1, e_2, e_3\} \\ &= \mathscr{L}\{e_1 + e_2 + e_3\} \oplus \mathscr{L}\left\{-\frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3, -\frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{2}{3}e_3\right\}. \end{aligned}$$

Now, let $\phi\colon G \to \mathrm{GL}_3(\mathbb{C})$ denote the representation given by left-multiplication with respect to the latter decomposition. We proceed to evaluate the expression $\phi(a) \in \mathrm{GL}_3(\mathbb{C})$. Left-multiplication by $a \in G$ does not affect the generating element $e_1 + e_2 + e_3$ in the linear span of $\{e_1 + e_2 + e_3\}$. Also, we have that:

$$\begin{aligned} a \cdot \left(-\frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3\right) &= -\frac{1}{3}e_2 + \frac{2}{3}e_3 - \frac{1}{3}e_1 \\ &= -\frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{2}{3}e_3. \end{aligned}$$

Now evaluate the expression given below:

$$\begin{aligned} a^2 \cdot \left(-\frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3\right) &= -\frac{1}{3}e_3 + \frac{2}{3}e_1 - \frac{1}{3}e_2 \\ &= \frac{2}{3}e_1 - \frac{1}{3}e_2 - \frac{1}{3}e_3 \\ &- \left(-\frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3\right) - \left(-\frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{2}{3}e_3\right). \end{aligned}$$

We may thus evaluate $\phi(a)$ as follows.

$$\phi(a) = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right]$$

Observe the block matrix structure of the above matrix. Evaluate $\phi(a^2)$ as follows.

$$\phi(a^2) = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

Recall that two elements in the same conjugacy class will have the same character value.

G	e	a	a^2
$\mathrm{tr}(\phi)$	3	0	0

The submodule $\mathscr{L}\{e_1 + e_2 + e_3\}$ is irreducible.

Question 12.1. Is the module $\mathscr{L}\{-\frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3, -\frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{2}{3}e_3\}$ irreducible?

Let $\chi^V = \mathrm{tr}\phi_V$.

Write $V = W_1 \oplus W_2$, with $W_1 = \mathscr{L}\{e_1 + e_2 + e_3\}$, and:

$$W_2 = \mathscr{L}\left\{-\frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3, -\frac{1}{3}e_1 - \frac{1}{3}e_2 + \frac{2}{3}e_3\right\}.$$

Now, consider the following character table:

G	e	a	a^2
$\chi^V = \mathrm{tr}\phi_V$	3	0	0
$\chi^{W_1} = \mathrm{tr}\phi_{W_1}$	1	1	1
$\chi^{W_2} = \mathrm{tr}\phi_{W_2}$	2	-1	-1

The above table nicely illustrates that

$$\chi^V(g) = \chi^{W_1}(g) + \chi^{W_2}(g)$$

for all $g \in G$.

12.2 Characters of regular representations

Let G be a finite group, and let $V = \mathscr{L}\{v_g : g \in G\}$ be such that $h \cdot v_g = v_{hg}$ for $h \in G$. Observe that:

$$\dim V = |G| = \chi^V(e) = \mathrm{tr}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{|G|\times|G|} = \mathrm{tr}(I_{|G|}).$$

Now, if $g \neq e$, we find that $\chi^V(g)$ is equal to the trace of a matrix of the form indicated below.

$$\chi^V(g) = \operatorname{tr} \begin{array}{r c} & \begin{matrix} v_e & v_{g_1} & & & & & & & \end{matrix} \\ \begin{matrix} \\ \\ \\ \\ v_g \\ \\ \\ v_{gg_1} \\ \\ \end{matrix} & \begin{bmatrix} 0 & 0 & & & & & & & \\ 0 & 0 & & & & & & & \\ \vdots & \vdots & & & & & & & \\ 0 & \vdots & & & & & & & \\ 1 & \vdots & & & & & & & \\ 0 & \vdots & & & & & & & \\ \vdots & 0 & & & & & & & \\ \vdots & 1 & & & & & & & \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{|G|\times|G|} \end{array} = 0.$$

Observe that $g \cdot v_e = v_g$ and $g \cdot v_{g_1} = v_{gg_1}$.

$$\chi^V(g) = \sum_{h\in G} \begin{cases} 1 & \text{if } g \cdot v_h = v_{gh} = v_h, \\ 0 & \text{otherwise.} \end{cases}$$

The matrices in this case are permutation matrices.

This representation is called the left regular representation.

12.3 Illustrations and applications of character theory

Recall that if two characters are equal, then the corresponding representations are isomorphic.

For example, using character theory, it is easily seen that the left regular representation is isomorphic to the right regular representation of a given group.

In the discrete Fourier transform, given data values $a_1, a_2, \ldots, a_\ell$ associated to $0, 1, \ldots, \ell - 1$, we form a correspondence

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_\ell \end{bmatrix} \cong \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_\ell \end{bmatrix}$$

whereby an expression of the form c_i represents the multiplicity of the ith irreducible representation of C_ℓ, the multiplicative cyclic group of order ℓ. We may decompose the former matrix with respect to the irreducible representations.

Now, let $G = D_3 = \{e, a, a^2, b, ba, ba^2\}$ denote the dihedral group of order 6.

Now consider the vector space

$$V = \mathscr{L}\{v_e, v_a, v_{a^2}, v_b, v_{ba}, v_{ba^2}\}$$

spanned by a set indexed by the underlying set of G.

We are going to act on this vector space indexed by elements elements D_3, but our action is not going to be left-multiplication. In particular, we are going to act on this vector space through conjugation:

$$g \bullet v_h = v_{ghg^{-1}}.$$

Now, observe that if we act on the vector space given above through conjugation, then $\mathscr{L}\{v_e\}$ is an invariant submodule.

Now, observe that V may be decomposed in the following manner:

$$\begin{aligned} V &= \mathscr{L}\{v_e, v_a, v_{a^2}, v_b, v_{ba}, v_{ba^2}\} \\ &= \mathscr{L}\{v_e\} \oplus \mathscr{L}\{v_a, v_{a^2}\} \oplus \mathscr{L}\{v_b, v_{ba}, v_{ba^2}\}. \end{aligned}$$

For an index i, let W_i denote the ith term in the above decomposition. Now, consider the following character table, and recall that characters are constant on conjugacy classes.

	e	a	a^2	b	ba	ba^2
χ^{W_1}	1	1	1	1	1	1
χ^{W_2}	2	2	2	0	0	0
χ^{W_3}	3	0	0	1	1	1

Since $bab^{-1} = a^2$, we have that a and a^2 are conjugate.

Since $a^{-2}ba^2 = ba$, and since $a \cdot ba \cdot a^2 = ba^2$, we find that b, ba, and ba^2 are all conjugate.

Observe that $a \cdot a \cdot a^{-1} = a$. So, when we act on a by χ^{W_2} and χ^{W_3}, they are fixed.

$$\begin{aligned} b \bullet v_b &= v_{bbb^{-1}} = v_b \\ b \bullet v_{ba} &= v_{bbab^{-1}} = v_{ab} = v_{ba^2} \\ b \bullet v_{ba^2} &= v_{b(ba^2)b^{-1}} = v_{ba}. \end{aligned}$$

With respect to the above equalities, we find that v_b is the only fixed point. So, the corresponding trace will be equal to 1, since v_b is the only fixed point.

How do we automatically know that $\chi^{W_3}_{ba} = 1$? We know that $\chi(g) = hgh^{-1}$. The character is equal on conjugacy classes. We thus obtain a character table of the following form.

	e	$\{a, a^2\}$	$\{b, ba, ba^2\}$
χ^{W_1}	1	1	1
χ^{W_2}	2	2	0
χ^{W_3}	3	0	1

12.4 A natural scalar product on characters

We define a scalar product $\langle \cdot, \cdot \rangle$ on characters as follows:

$$\begin{aligned} \langle \chi, \psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}. \end{aligned}$$

We may assume without loss of generality that the matrices under consideration are all unitary: $A^{-1} = \overline{A^T}$.

All representations are unitary because we assumed we are working over $\mathbb{C}$ in finite-dimensional vector spaces and finite groups.

Intuitively, it is often easier to work with inverses instead of conjugates.

Theorem 12.2. *If M and N are irreducible, then:*

$$\boxed{\langle \chi^M, \chi^N \rangle = \begin{cases} 1 & \text{if } M \cong N, \\ 0 & \text{otherwise.} \end{cases}} \tag{12.1}$$

For example, consider the vector space $\mathscr{L}\{v_b, v_{ba}, v_{ba^2}\}$. Does the sum of the elements in the generating set $\{v_b, v_{ba}, v_{ba^2}\}$ give an irreducible representation?

Note: $\mathscr{L}\{v_a + v_{a^2}\}$ is a submodule of $\mathscr{L}\{v_a, v_{a^2}\}$.

Also observe that $\mathscr{L}\{v_b + v_{ba} + v_{ba^2}\}$ is a submodule of $\mathscr{L}\{v_b, v_{ba}, v_{ba^2}\}$.

There has to be an orthogonal complement to $\mathscr{L}\{v_a + v_{a^2}\}$ and $\mathscr{L}\{v_b + v_{ba} + v_{ba^2}\}$. If it has an orthogonal complement, it has to break down into smaller ones.

$$\begin{aligned}
\langle \chi^{W_1}, \chi^{W_1} \rangle &= \frac{1}{6}\left(\chi^{W_1}(e)\chi^{W_1}(e^{-1}) + \chi^{W_1}(a)\chi^{W_1}(a^2) + \cdots\right) \\
&= \frac{1}{6}(1+1+1+1+1+1) \\
&= \frac{1}{6}(6) \\
&= 1.
\end{aligned}$$

Now, evaluate the expression $\langle \chi^{W_2}, \chi^{W_2} \rangle$.

$$\begin{aligned}
\langle \chi^{W_2}, \chi^{W_2} \rangle &= \frac{1}{6}\left(2^2 + 2^2 + 2^2 + 0 + 0 + 0\right) \\
&= \frac{12}{6} = 2.
\end{aligned}$$

"Why" is $\langle \chi^{W_2}, \chi^{W_2} \rangle$ equal to 2? What is this telling us? What does this represent?

Let X be a matrix of indeterminates: $[x_{ij}]_{m\times n} = X$. Let A be a representation corresponding to M, given a fixed basis, and let V be the representation corresponding to N.

So, $A(g)$ is an $m \times m$ invertible matrix such that $\dim(M) = m$, and $B(g)$ is an $n \times n$ invertible matrix with $\dim(N) = n$. Define

$$Y = \frac{1}{|G|}\sum_{g\in G} A(G)XB(g^{-1}),$$

where $X = [x_g]_{m\times n}$. Similarly,

$$A(h)YB(h^{-1}) = \frac{1}{|G|}\sum_{g\in G} A(h)A(g)XB(g^{-1})B(h^{-1}).$$

Since $A(h)Y = YB(h)$, we have that Y is a G-homomorphism from M to N. This is true for any indeterminate matrix X.

If this is a G-homomorphism from M to N and M and N are irreducibles, what can be said about Y?

By Schur's Lemma, since M and N are irreducible, $Y = 0$, or $Y = c\mathrm{Id}_{m\times m}$ and $M = N$.

If $M \ncong N$, then $Y = 0$, and:

$$\langle \chi^M, \chi^N \rangle \stackrel{?}{=} 0.$$

Look at the i, j entry of

$$Y = \frac{1}{|G|} \sum_{g\in G} \sum_{k=1}^{m} \sum_{\ell=1}^{n} A_{i,k}(g) X_{k,\ell} B_{\ell,j}(g^{-1}) = 0.$$

In particular, this is true no matter what the indeterminates are. This is equal to 0 for any k, ℓ.

$$0 = \frac{1}{|G|} \sum_{g\in G} A_{ik}(g) B_{\ell,j}(g^{-1}).$$

Since this is true for all indeterminate matrices X, if we fix values for X, we obtain the above equation.

$$\begin{aligned}
\langle \chi^M, \chi^N \rangle &= \frac{1}{|G|} \sum_{g\in G} \chi^M(g) \chi^N(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g\in G} \sum_{i=1}^{m} A_{ii}(g) \sum_{j=1}^{n} B_{jj}(g^{-1}) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{1}{|G|} \sum_{g\in G} A_{ii}(g) B_{jj}(g^{-1}) \\
&= 0.
\end{aligned}$$

If $M \cong N$, we can use essentially the same technique.

Now, let $V = \oplus_{i=1}^{k} V_i^{\oplus m_i}$, where m_i denotes the multiplicity of V_i in V.

Theorem 12.3. $\chi^V = \sum_{i=1}^{k} m_i \chi^{V_i}$.

Theorem 12.4. $\langle \chi^V, \chi^{V_i} \rangle = \sum_{j=1}^{k} \langle m_j \chi^{V_j}, \chi^{V_i} \rangle = m_i$.

Theorem 12.5. $\langle \chi^V, \chi^V \rangle = \sum_{j=1}^{k} \sum_{i=1}^{k} \langle m_j \chi^{V_j}, m_i \chi^{V_i} \rangle = \sum_{i=1}^{k} m_i^2$.

Theorem 12.6. *The scalar product $\langle \chi^V, \chi^V \rangle$ is equal to 1 iff exactly one expression of the form m_i is equal to 1 and all the others are equal to 0 iff V is irreducible.*

Example 12.7. Recall that $\langle \chi^{W_2}, \chi^{W_2} \rangle = 2$, as above. Therefore, χ^{W_2} is not irreducible.

Theorem 12.8. *If $\chi^M = \chi^V$, then $M \cong V$, and:*

$$\langle \chi^M, \chi^{V_i} \rangle = m_i \Longrightarrow M \cong \oplus_{i=1}^{k} V_i^{\oplus m_i} \cong V.$$

Note that the converse also holds: the above theorem is biconditional.

We thus have that characters determine the representations.