PRACTICE FOR MATH 6121 MIDTERM

OCTOBER 23, 2016

Try and answer 2 of the following 4 questions completely in 80 minutes.

(1) Let G be a finite group, and let $Z(G)$ be the center of G.
  (a) Show that the order of $G/Z(G)$ is not a prime.
  (b) Suppose that $|G| = 175$. Show that G is solvable and give its composition factors.

(2) Let G be a finite group and let $\gamma : G \to Aut(G)$ be defined by $\gamma(g)(h) = ghg^{-1}$. We can now define the semidirect product $G \rtimes_\gamma G$ as the group consisting of the set of pairs $\{(g_1, g_2) : g_1, g_2 \in G\}$ with the product $(g_1, g_2) \cdot_{G \rtimes_\gamma G} (g_3, g_4) = (g_1\gamma(g_2)(g_3), g_2g_4)$.
  (a) Show that $H = \{(g, 1) : g \in G\} \triangleleft (G \rtimes_\gamma G)$.
  (b) Show that $K = \{(g, g^{-1}) : g \in G\} \triangleleft (G \rtimes_\gamma G)$.
  (c) Show that $(G \rtimes_\gamma G) \cong G \times G$

(3) Let $G = \{e, \gamma, \gamma^2, \ldots, \gamma^{n-1}\}$ with $\gamma^n = e$ be the cyclic group of order n. Fix an integer d, let $M_d = \mathbb{C}$ be a vector space of dimension 1. Let G act on $z \in M_d$ with the action $\gamma.z = \zeta^d z$ where $\zeta = e^{2\pi i/n}$ is a primitive n^{th} root of unity. [Here you have that $\gamma^2.z = \gamma.(\gamma.z) = \zeta^{2d}z$ and similarly $\gamma^k.z = \zeta^{kd}z$.]
  (a) For which integers d is M_d a G-module?
  (b) For which integers d is M_d irreducible?
  (c) When is $M_d \cong M_{d'}$?

(4) Let X be a G-set for a finite group G. We denote by

$$X/G = \{\{g.x : g \in G\} : x \in X\}.$$

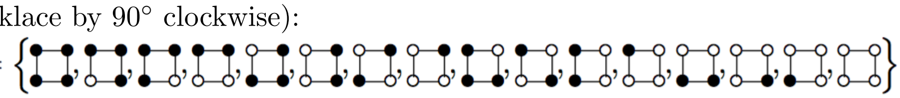
That is, X/G is the set of equivalence classes of X via the action of G. In other words, X/G is the set of orbits of the action of G on X. For $g \in G$, let

$$Fix_X(g) = \{x \in X : g.x = x\}\ .$$

  (a) Show that

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |Fix_X(g)|\ .$$

  Hint: count the cardinality of $\{(g, x) : g \in G, x \in X, g.x = x\}$ in two different ways
  (b) Let $G = \mathbb{Z}_4 = \{e, \gamma, \gamma^2, \gamma^3\}$ with $\gamma^4 = e$ be the cyclic group of order 4. Let G act by rotation on the set of 4-necklaces with black and white beads (where $\gamma.N$ is a rotation of the necklace by 90° clockwise):

$X =$ 

  Describe the set $Fix_X(g)$ for each element $g \in \mathbb{Z}_4$.
  (c) Use (a) and (b) to count the number of different 4-necklaces up to the action of $G = \mathbb{Z}_4$.