Polynomial Rings

All rings in this note are commutative.

Proposition: The polynomial ring $R[x]$ is a Principal Ideal Domain if and only if the ring R is a field.
Proposition: The polynomial ring $R[x]$ is a Euclidean Domain $\Longrightarrow (b(x)) = (a_1(x), a_2(x), \cdots, a_n(x))$

Example:

$$Q[x] = (1) = (x^3+1, x^2+x+1) = \{p(x)(x^3+1)+q(x)(x^2+x+1) \text{ for some polynomials } p(x) \text{ and } q(x)\}$$

$$x^3 + 1 = (x^2 + x + 1)(x - 1) + 2 \cdot 1$$

$$1 = \frac{1}{2}(x^3 + 1) - \frac{1}{2}(x^2 + x + 1)(x - 1)$$

$$\Longrightarrow 1 \in (x^3 + 1, x^2 + x + 1) \Longrightarrow p(x) \in (x^3 + 1, x^2 + x + 1) \Longrightarrow Q[x] = (x^3 + 1, x^2 + x + 1)$$

Proposition: If the ideal I is a maximal ideal then the quotient ring R/I is a field.

$$I \subseteq R \qquad I/I \subseteq R/I$$

Proposition: If the ideal I is a prime ideal in R then the quotient ring R/I is an integral domain.