MORE PRACTICE FINAL

(1) Assume that the following diagram of group homomorphisms commutes and that the two rows are exact sequences.

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\phi} & C \\ \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\phi'} & C' \end{array}$$

Prove that if β is surjective and if γ and ψ' are injective then α is surjective.

(2) Let I and J be ideals of R, a ring with identity $1 \neq 0$. Recall that IJ is the set of all finite sums of products xy where $x \in I$ and $y \in J$.
  (a) Prove that $I + J$ is the smallest ideal containing both I and J.
  (b) Prove that IJ is an ideal contained in $I \cap J$.
  (c) Give an example where $IJ \neq I \cap J$.
  (d) Prove that if R is commutative and if $I + J = R$, then $IJ = I \cap J$

(3) Let A and B be ideals in R, a commutative ring with $1 \neq 0$. For $r \in R$, prove that $\phi(r) = (r+A, r+B)$ is a ring homomorphism from R to $R/A \times R/B$ and compute $ker\ \phi$.

(4) Let $\mathbb{F}$ be a field. Show that the subring $\mathbb{F}[x, x^2y, x^3y^2, \ldots, x^ny^{n-1}, \ldots]$ of the ring $\mathbb{F}[x, y]$ contains an ideal which is not finitely generated.

(5) Prove that the ring $\mathbb{Z}[x_1, x_2, x_3, \ldots]/\langle x_1x_2, x_3x_4, x_5x_6, \ldots\rangle$ contains infinitely many minimal prime ideals.

(6) Suppose that I is a monomial ideal generated by monomials $m_1, m_2, \ldots, m_k$.
  (a) Prove that the polynomial $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is in I if and only if every monomial term f_i of f is a multiple of one of the m_j.
  (b) Fix a monomial ordering on $R = \mathbb{F}[x_1, x_2, \ldots, x_n]$ and suppose that $\{g_1, g_2, \ldots, g_m\}$ is a Gröbner basis for the ideal I in R. Prove that $h \in LT(I)$ if and only if h is a sum of monomial terms each of which is divisible by some $LT(g_i)$.

(7) Show that $\{x - y^3, y^5 - y^6\}$ is a Gröbner basis for the ideal $I = \langle x - y^3, -x^2 + xy^2\rangle$ with respect to lexicographic ordering where $x > y$ in the ring $\mathbb{F}[x, y]$.

(8) Prove that the rings $\mathbb{F}[x, y]/\langle y^2 - x\rangle$ and $\mathbb{F}[x, y]/\langle y^2 - x^2\rangle$ are not isomorphic for any field $\mathbb{F}$.