A BIG EXAMPLE SOLUTIONS - MATH 6161

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(1) (Sandeep) Lets denote the conjugacy class of an element $x \in G$ as x^G. From now on $G = D_n$. For n odd: Since $e \in Z(D_n)$ $e^{D_n} = \{e\}$, $\{x^i, x^{-i}\}$ for $1 \leq i \leq n-1$ and $\{y, xy, x^2y, \ldots, x^{n-1}y\}$ are all of the conjugacy classes. There are $1 + 1 + (n-1)/2 = (n+3)/2$ classes in total.
For n even $= 2m$: Again we have $\{e\}$ as well $\{x^m\}$ and $\{x^i, x^{-i}\}$ for $1 \leq i \leq m-1$, also $\{y, x^2y, \ldots, x^{n-2}y\}$, $\{xy, x^3y, \ldots, x^{n-1}y\}$ are all of the conjugacy classes. There are $1 + 1 + 1 + 1 + (n-2)/2 = (n+6)/2 = m+3$ classes in total.
Why: e in center, $(x^jy)x^i(x^jy)^{-1} = x^{-i}$ so x^i and x^{-i} are conjugate, $(x^jy)x^iy(x^jy)^{-1} = x^{2j-i}$. If n is even then $2j - i$ will always be even if i is even and will always be odd if i is odd so there are two conjugacy classes when n is even. If n is odd then $2j - i$ will can take on any value for i fixed.

(2) (Ziting/Weihong) We must have that $\chi(x) = a$ and $\chi(y) = b$ and it must satisfy the relations $a^n = 1$, $a^2 = 1$, $b^2 = 1$ because $1 = \chi(e) = \chi(x^n) = a^n$, x is conjugate to x^{-1} so $a = \chi(x) = \chi(x^{-1}) = a^{-1}$, and $1 = \chi(e) = \chi(y^2) = \chi(y)^2 = b^2$. If n is even then we have that $a^2 = 1$ and $b^2 = 1$ characterizes these values and there are exactly 4 1-dimensional representations of D_n (with $\chi(x) = \pm 1$ and $\chi(y) = \pm 1$). If n is odd then $a = 1$ and $b^2 = 1$ characterizes the relations and there are exactly 2 1-dimensional representations (with $\chi(x) = 1$ and $\chi(y) = \pm 1$).

(3) (Ziting) Let m_i be the dimensions of the irreducibles in D_n. We know that $\sum m_i^2 = |D_n| = 2n$.
If n is odd then there are $(n+3)/2$ conjugacy classes and two 1-dimensional representations so the remaining $(n-1)/2$ representations have dimension greater $m_i \geq 2$. $\sum m_i^2 = 2 + \sum_{\text{not 1-d reps}} m_i^2 = 2n$ so we have the sum $\sum_{\text{not 1-d reps}} m_i^2 = 2(n-1)$ and since there are $(n+3)/2$ irreducible representations in total and $(n+3)/2 - 2 = (n-1)/2$ are not one dimensional, then

$$\sum_{\text{not 1-d reps}} m_i^2 \geq 2^2 \text{ (\# of not 1-d reps)} = 2(n-1)$$

with equality if and only if all of the $m_i = 2$. Therefore all $m_i = 2$.
A similar argument works for n even. We have $\sum m_i^2 = 4 + \sum_{\text{not 1-d reps}} m_i^2 = 2n$, so that $\sum_{\text{not 1-d reps}} m_i^2 = 2(n-2)$. We also know that there are $(n+6)/3$ irreducible representations in total and $(n+6)/2 - 4 = (n-2)/2$ which are not 1-dimensional, then we have

$$\sum_{\text{not 1-d reps}} m_i^2 \geq 2^2 \text{ (\# of not 1-d reps)} = 2n - 4$$

with equality if an only if all of the $m_i = 2$ (and we do have equality).

(4) (Weihong) For D_4, list the 1-dimensional representations and the last one can be obtained by orthogonality relations.

D_4	e	x^2	x, x^3	y, x^2y	xy, x^3y
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	1	1	-1	1	-1
$\chi^{(4)}$	1	1	-1	-1	1
$\chi^{(5)}$	2	-2	0	0	0

For D_5 the table is 4×4 with the first two modules are 1-dimensional and the last two are two dimensional.

D_5	e	x, x^4	x^2, x^3	y, xyx^2y, x^3y, x^4y
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	1	1	-1
$\chi^{(3)}$	2	a	b	c
$\chi^{(4)}$	2	d	e	f

Use the character relations between the rows to solve for the remaining entries.

$$2 + 2a + 2b + 5c = 0, 2 + 2a + 2b - 5c = 0, 2 + 2d + 2e + 5f = 0, 2 + 2d + 2e - 5f = 0$$

this implies that $c = 0$, $f = 0$, $a = e$ and $b = d = 1 - a$ In addition by the orthogonality relations, $\frac{1}{10}(2^2 + 2a^2 + 2b^2) = 1$ and so $a = \frac{-1 \pm \sqrt{5}}{2}$.

D_5	e	x, x^4	x^2, x^3	y, xyx^2y, x^3y, x^4y
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	1	1	-1
$\chi^{(3)}$	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0
$\chi^{(4)}$	2	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0

(5) (Ziting) Notice that $a_{1-i} = a_{i+1 \ mod \ 2}$ so we can say that $x^j(a_i) = a_{i+j \ mod \ 2}$. Now compute the action on a basis to show that $g(h(v)) = (gh)(v)$. Remark: You might only need to show that the group relations are satisfied $x^4(v) = v$, $y^2(v) = v$, $x(y(v)) = y(x^{-1}(v))$, but you would have to justify that this is sufficient. Instead do the following 8 calculations:

$$(x^j x^k)(a_i) = a_{i+j+k \ mod2} = x^j(x^k(a_i)) \qquad (x^j x^k)(b_i) = b_{i+j+k \ mod2}$$

$$(x^j x^k y)(a_i) = b_{i+j+k \ mod2} = x^j((x^k y)(a_i)) \qquad (x^j x^k y)(b_i) = a_{i+j+k \ mod2} = x^j((x^k y)(b_i))$$

$$(x^j y x^k)(a_i) = b_{i+j-k \ mod2} = (x^j y)(x^k(a_i)) \qquad (x^j y x^k)(b_i) = a_{i+j-k \ mod2} = (x^j y)(x^k(b_i))$$

$$(x^j y x^k y)(a_i) = a_{i+j-k \ mod2} = (x^j y)((x^k y)(a_i)) \qquad (x^j y x^k y)(b_i) = b_{i+j-k \ mod2} = (x^j y)((x^k y)(b_i))$$

(6) (Huilan) $V = \mathcal{L}\{a_0, a_1, b_0, b_1\}$, $X : D_4 \to Gl(V)$ by

$$X(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = X(x^2)$$

$$X(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = X(x^3)$$

$$X(y) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = X(x^2y)$$

$$X(xy) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = X(x^3y)$$

The character is then $\chi(e) = \chi(x^2) = 4$, $\chi(a) = 0$ for $a \neq e, x^2$.

(7) (Huilan) Computing the scalar products in the character table we find that

$$\left\langle \chi, \chi^{(i)} \right\rangle = \frac{1}{8}(4 + 4 + 0 + 0 + 0) = 1$$

and so we conclude that $\chi = \chi^{(1)} + \chi^{(2)} + \chi^{(3)} + \chi^{(4)}$ which is the sum of all of the 1-dimensional representations.

(8) (Dan) Either guess or use orthogonality relations to determine the following subspaces.

$$\mathcal{L}\{a_0, a_1, b_0, b_1\} = \mathcal{L}\{a_0 + a_1 + b_0 + b_1\} \oplus \mathcal{L}\{a_0 + a_1 - b_0 - b_1\} \\ \oplus \mathcal{L}\{-a_0 + a_1 + b_0 - b_1\} \oplus \mathcal{L}\{a_0 - a_1 + b_0 - b_1\}$$

The spaces are listed in order, such that each corresponds to $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$ and $\chi^{(4)}$ respectively. Check the action of the generators on these in order to show that they agree with the character table.

$$x(a_0 + a_1 + b_0 + b_1) = a_0 + a_1 + b_0 + b_1$$
$$y(a_0 + a_1 + b_0 + b_1) = a_0 + a_1 + b_0 + b_1$$

$$x(a_0 + a_1 - b_0 - b_1) = a_0 + a_1 - b_0 - b_1$$
$$y(a_0 + a_1 - b_0 - b_1) = -(a_0 + a_1 - b_0 - b_1)$$

$$x(-a_0 + a_1 + b_0 - b_1) = -(-a_0 + a_1 + b_0 - b_1)$$
$$y(-a_0 + a_1 + b_0 - b_1) = -(-a_0 + a_1 + b_0 - b_1)$$

$$x(a_0 - a_1 + b_0 - b_1) = -(a_0 - a_1 + b_0 - b_1)$$
$$y(a_0 - a_1 + b_0 - b_1) = a_0 - a_1 + b_0 - b_1$$

(9) (Sandeep) There is an isomorphism between monomials of degree k and sequences of k dots and 3 bars. A $a_0^{r_1}a_1^{r_2}b_0^{r_3}b_1^{r_4}$ is associated to r_1 dots followed by a bar, r_2 dots followed by a bar, r_3 dots followed by a bar and r_4 dots. There are $\binom{k+3}{3}$ different sequences of this types and so there are the same number of monomials.

(10) (Dan) The action on the polynomials is the one induced from the action of x and y on the variables a_0, a_1, b_0, b_1. We define $x(a_0^{r_1}a_1^{r_2}b_0^{r_3}b_1^{r_4}) = a_0^{r_2}a_1^{r_1}b_0^{r_4}b_1^{r_3}$ and $y(a_0^{r_1}a_1^{r_2}b_0^{r_3}b_1^{r_4}) = a_0^{r_3}a_1^{r_4}b_0^{r_1}b_1^{r_2}$. Notice that for $g \in D_4$ we have $g(a_0^{r_1}a_1^{r_2}b_0^{r_3}b_1^{r_4}) = g(a_0)^{r_1}g(a_1)^{r_2}g(b_0)^{r_3}g(b_1)^{r_4}$ and we already showed that g acting on each of the variables defines a D_4-homomorphism, therefore this action defines a D_4-homomorphism on the monomials in these variables.

(11) (Marcus) Note that by (9), P_2 has 10 basis elements. They are a_0^2, a_1^2, b_0^2, b_1^2, a_0a_1, a_0b_0, a_0b_1, a_1b_0, a_1b_1, and b_0b_1. Calculating the action of a representative of each of the five equivalence classes of D_4 on the basis elements and noting the number of the elements of the basis kept fixed, we get $\chi(e) = 10$, $\chi(x^2) = 10$, $\chi(x) = \chi(x^3) = 2$, $\chi(y) = \chi(x^2y) = 2$, and $\chi(xy) = \chi(x^3y) = 2$.

(12) (Marcus) Calculating the inner product of χ with the irreducible characters of D_4 [see the solution to (4)] will give us the multiplicities of each of the irreducibles of P_2. Calculating we get $< \chi, \chi^{(1)} >= 4$, $< \chi, \chi^{(2)} >= 2$, $< \chi, \chi^{(3)} >= 2$, $< \chi, \chi^{(4)} >= 2$, and $< \chi, \chi^{(5)} >= 0$. As a check, note that

$$\chi(e) = 10 = 4\chi^{(1)}(e) + 2\chi^{(2)}(e) + 2\chi^{(3)}(e) + 2\chi^{(4)}(e) + 0\chi^{(5)}(e).$$