FINAL EXAM - MATH 6161

JUNE 12, 2003

PART I: written and computational. Instructions: Do any 4 of the following 6 problems.

(1) Find the value of $\langle p_1^n, h_k h_{n-k}\rangle$. Use this to compute the dimension of the irreducible character χ^μ for μ a two row partition.
(2) Expand $h_{(3,2,1)}$ in the
  (a) p-basis
  (b) e-basis
  (c) s-basis
(3) Use the following identity,

$$\Delta(p_\mu) = \sum_\lambda \frac{p_\lambda}{z_\lambda} \otimes \left(p_\lambda^\perp p_\mu\right)$$

to show in general that for any dual bases $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$, and for any $f \in \Lambda$,

$$\Delta(f) = \sum_\lambda a_\lambda \otimes \left(b_\lambda^\perp f\right).$$

(4) Calculate $\left\langle h_{(3,3)}, h_{(3,2,1)}\right\rangle$, or equivalently, find the coefficient of $m_{(3,3)}$ in $h_{(3,2,1)}$.
(5) Determine the coefficient of z^0, z^1, z^2, z^3 and z^4 in the expression $m_{(3,2,1)}[X+z]$.
(6) You are given below a table of coefficients of p_λ/z_λ in h_μ (μ indexes the left side of the table and λ the row across the top). Use this to calculate the first 6 rows of the character table for S_6. Explain in a few words how you can easily find the last 5 rows from the first 5.

	(1^6)	$(2,1^4)$	$(2^2,1^2)$	$(3,1^3)$	(2^3)	$(3,2,1)$	$(4,1^2)$	(3^2)	$(4,2)$	$(5,1)$	(6)
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$	6	4	2	3	0	1	2	0	0	1	0
$(4,2)$	15	7	3	3	3	1	1	0	1	0	0
$(4,1,1)$	30	12	2	6	0	0	2	0	0	0	0
$(3,3)$	20	8	4	2	0	2	0	2	0	0	0
$(3,2,1)$	60	16	4	3	0	1	0	0	0	0	0
$(3,1,1,1)$	120	24	0	6	0	0	0	0	0	0	0
$(2,2,2)$	90	18	6	0	6	0	0	0	0	0	0
$(2,2,1,1)$	180	24	4	0	0	0	0	0	0	0	0
$(2,1,1,1,1)$	360	24	0	0	0	0	0	0	0	0	0
$(1,1,1,1,1,1)$	720	0	0	0	0	0	0	0	0	0	0

PART II: We will do these problems together during the 2nd and 3rd hours. These problems are all interconnected and a mistake on one will make the others impossible to do. Each person in the class will be responsible for a single problem, but everyone is to help out. Failure to do so will result in a lower grade for this part of the exam.

Set

$$\Delta(x_1, x_2, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j)$$

Let T be an injective tableau of shape $\lambda \vdash n$ (that is $T : D(\lambda) \to \{1, 2, \ldots, n\}$ is an injective map). Define $G_T(X_n)$ to be the polynomial

$$\Delta(x_{T_{(1,1)}}, x_{T_{(2,1)}}, \ldots, x_{T_{(\lambda'_1,1)}})\Delta(x_{T_{(1,2)}}, x_{T_{(2,2)}}, \ldots, x_{T_{(\lambda'_2,2)}}) \cdots \Delta(x_{T_{(1,\lambda_1)}}, x_{T_{(2,\lambda_1)}}, \ldots, x_{T_{(\lambda'_{\lambda_1},\lambda_1)}})$$

(e.g. if T is the tableau

$$\begin{array}{|c|c|c|}\hline 7 & 8 & \\ \hline 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

then $G_T(X_9) = \Delta(x_1, x_4, x_7)\Delta(x_2, x_5, x_8)\Delta(x_3, x_6)$). In particular, let $G_\lambda(X_n)$ be the polynomial associated to the tableau which has the numbers 1 through λ'_1 in the first column, $\lambda'_1 + 1$ through $\lambda'_1 + \lambda'_2$ in the second column, etc.

Let V^λ be the S_n module spanned by all of the polynomials $G_T(X_n)$ for T an injective tableau of shape λ.

1. Show that for any partition λ, if $\pi \in S_{\lambda'_1} \times S_{\lambda'_2} \times \cdots \times S_{\lambda'_{\lambda_1}} \subset S_n$ then $\pi G_\lambda(X_n) = \epsilon(\pi) G_\lambda(X_n)$. Show that any polynomial with this property is divisible by $G_\lambda(X_n)$.
2. Show that $V^{(2,2)}$ is spanned by the $G_T(X_4)$ where T is a standard tableau of shape $(2, 2)$ and that they are linearly independent.
3. Compute the S_4-character of this module. Show that it is irreducible.
4. Give a basis for $V^{(2,2)} \otimes V^{(2,2)}$. Give the S_4 character when S_4 acts on it by the action $\pi(v \otimes w) = (\pi v) \otimes (\pi w)$. Break down this module into irreducible components.
5. See the definition of the induced submodule (below). Give some sort of representation for $V^{(2,2)}\uparrow_{S_4}^{S_5}$ and give a basis with a definition of the action of S_5 on this basis. How does this module differ from $V^{(2,2,1)}$?
6. Compute the character of this module and give a decomposition of the character into a sum of irreducible characters.
7. Compute the character of the S_4 module consisting of all products of uv, for $u, v \in V^{(2,2)}$. How does this differ from the module $V^{(2,2)} \otimes V^{(2,2)}$ in problem (4)?

If V is an H module with $H \subseteq G$ then the induced module V from H to G, $Ind\ V\uparrow_H^G$, is the space $(\mathbb{Q}G \otimes V)/W$ where W is the subspace linearly spanned by the elements of the form $(gh) \otimes v - g \otimes (hv)$ for $g \in G$, $h \in H$ and $v \in V$.