HOMEWORK PROBLEMS - MATH 6161

HOMEWORK # 1 ON PARTITIONS AND SYMMETRIC FUNCTIONS

(1) In the exam we had a family of D_4-modules P_k for $k \geq 0$ which were the polynomials of degree k in the variables a_0, a_1, b_0, b_1. In addition we constructed the character table for D_4 and found that there were four 1-dimensional characters and a fifth character $\chi^{(5)}$ that was 2-dimensional. Show that

$$\left\langle P_k, \chi^{(5)} \right\rangle = 0$$

for all $k > 0$.

(2) The union of two partitions $\lambda \cup \mu$ is the smallest partition whose diagram contains the diagram of both λ and μ. Show that the union of all partitions of size n is a partition of size $\sum_{k=1}^{n} d(k)$ where $d(k)$ is the number of divisors of k.

(3) The dominance order is defined to be $\lambda < \mu$ if and only if $\lambda_1+\lambda_2+\cdots+\lambda_i < \mu_1+\mu_2+\cdots+\mu_i$ for all i. Show that $\lambda < \mu$ if and only if $\mu' < \lambda'$.

(4) The q-binomial coefficient is defined

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$$

where $[n]_q = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. Show

$$e_k \left[\frac{1-q^n}{1-q} \right] = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

and

$$h_k \left[\frac{1-q^n}{1-q} \right] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$$

using the recurrence $e_k[X+z] = e_k[X] + z e_{k-1}[X]$ and $h_k[X+z] = \sum_{i=0}^{k} z^i h_{k-i}[X]$.

(5) Use the computer to calculate the following scalar products
   (a) $\left\langle h_{(2,2,1)}, p_{(3,2)} \right\rangle$
   (b) $\left\langle h_{(3,2)}, p_{(3,2)} \right\rangle$
   (c) $\left\langle h_{(3,2)}, p_{(2,2,1)} \right\rangle$
   (d) $\left\langle h_{(3,2)}, h_{(4,1)} \right\rangle$
   (e) $\left\langle h_{(3,2)}, h_{(3,1,1)} \right\rangle$
   (f) $\left\langle h_{(3,2)}, h_{(2,2,1)} \right\rangle$

(6) Either use the computer to conjecture the following formulas or prove them. They may all be calculated directly using formulas in chapter 1.
   (a) $\langle h_n, p_\lambda \rangle$
   (b) $\langle e_n, p_\lambda \rangle$
   (c) $\langle p_n, h_\lambda \rangle$
   (d) $\langle p_{1^n}, h_\lambda \rangle$
   (e) $\langle p_\lambda, h_\lambda \rangle$
   (f) $\langle h_n, h_n \rangle$
   (g) $\langle e_n, h_n \rangle$

(h) $\langle h_n, h_\lambda \rangle$
(i) $\langle e_n, h_\lambda \rangle$

(7) Make a 5×5 table with labels $p_\mu, h_\mu, e_\mu, m_\mu$ and f_μ along the left of the table and $p_\lambda, h_\lambda, e_\lambda, m_\lambda$ and f_λ along the top of the table. Fill the entry with a_μ on the left and b_λ on the top with the coefficient of b_λ in a_μ as it is given in the text handed to you. Which entries were not discussed? There should be 4 which were not given a name in the text. Label these entries $F_{\lambda\mu}, G_{\lambda\mu}, H_{\lambda\mu}, I_{\lambda\mu}$. Find some sort of formula for these coefficients which allows you to determine the sign of the entry which is dependent on $\ell(\lambda)$, $\ell(\mu)$ and n (the sizes of the partitions λ and μ).

(8) Show that $\langle e_\lambda, h_\mu \rangle \neq 0$ if and only if $\lambda' \geq \mu$ and $\langle e_{\mu'}, h_\mu \rangle = 1$.

(9) Use Maple to create the transition matrix $A = [A_{\lambda\mu}]$ with $\lambda, \mu \vdash 6$. You may enter this by hand, but you might also want to use the functions

```
[seq(coeff( mexpr, m[op(mu)]), mu = Par(6))];
```

where `mexpr` is an expression in the monomial basis of degree 6 (in this case you want it to be `tom(` h_λ `)`). This matrix should have the property that $A\,\overrightarrow{m} = \overrightarrow{h}$ where $\overrightarrow{m}$ is a column vector. Verify this with the command:

```
map(toh,evalm( A&*[seq([m[op(la)]],la=Par(6))]));
```

What do the entries of this matrix represent in terms of scalar products? Observe and prove the following two properties of this matrix: 1. the entries are positive integers and 2. $A = A^t$. Use Maple to compute the LU-decomposition of this matrix (that is find matricies L, which is lower triangular, and U, which is upper triangular, such that $A = LU$) What do the entries of L and U represent? (see `LUdecomp` in the `linalg` package). Prove that $U = L^t$. What does the column vector $U\overrightarrow{m} = L^{-1}\overrightarrow{h}$ represent?