MIDTERM : MATH 6161

MAY 28, 2014

I want to work through some examples of the theorems that we have been talking about for the last weeks. I expect this exam will take a while and I will reserve 1/2 hour for each question. If the problem takes more than a half hour you will be responsible to finish it at home.

Here are the rules of the game:

(1) You will be assigned some of the questions below by draw of a number.
(2) You will be asked to write the answer on the blackboard for the problem that you draw.
(3) In addition, you will be able to ask for help (especially from your random partner).
(4) Write as clearly and legibly as you can and try to be consistent with notation from problem to problem.
(5) I will record the answer and make it available for next time.
(6) I will have my computer with me to do calculations and you may ask me to do heavy calculations that may speed up the process.

(1) Let ψ and ϕ be two irreducible characters of G representations.
   (a) Prove that if $\psi(e) = 1$, then $\tau(g) := \psi(g)\phi(g)$ is an irreducible character.
   (b) For $G = S_4$ show that $\chi^{(1)}(g) = 1$ and $\chi^{(2)}(id) = \chi^{(2)}((123)) = \chi^{(2)}((12)(34)) = 1$ and $\chi^{(2)}((12)) = \chi^{(2)}((1234)) = -1$ determine two 1 dimensional characters.
   (c) Show that if $\sigma(x_i) = x_{\sigma(i)}$ for $1 \leq i \leq 4$, then the vector space $\mathcal{L}\{x_1 - x_2, x_2 - x_3, x_3 - x_4\}$ is an irreducible 3-dimensional module and give its character (call it $\chi^{(3)}$).
   (d) Use part (a) to show that $\chi^{(4)}(\sigma) := \chi^{(2)}(\sigma)\chi^{(3)}(\sigma)$ is a third irreducible character.
   (e) Use the orthogonality relations to determine the last irreducible character and give the complete character table.

(2) The previous problem defines the character table of S_4. The subgroup A_4 of S_4 consists of the 'even' permutations. Let $C_1 = \{(1)\}$, $C_2 = \{(12)(34), (13)(24), (14)(23)\}$, $C_3 = \{(134), (142), (123), (243)\}$, and $C_4 = \{(143), (132), (124), (234)\}$. The group A_4 decomposes into conjugacy classes

$$A_4 = C_1 \cup C_2 \cup C_3 \cup C_4 \ .$$

   (a) Show that $\chi^{(1)}\downarrow^{S_4}_{A_4} = \chi^{(2)}\downarrow^{S_4}_{A_4}$ and $\chi^{(3)}\downarrow^{S_4}_{A_4} = \chi^{(4)}\downarrow^{S_4}_{A_4}$, that is, they agree as functions for all elements of A_4.
   (b) Show that $\chi^{(1)}\downarrow^{S_4}_{A_4}$ and $\chi^{(3)}\downarrow^{S_4}_{A_4}$ are irreducible in A_4.
   (c) Show that $\chi^{(5)}\downarrow^{S_4}_{A_4}$ is not irreducible.
   (d) Find two A_4 characters χ and χ' which are irreducible such that $\chi^{(5)}\downarrow^{S_4}_{A_4} = \chi + \chi'$. Give the character table of A_4.

(3) The group A_4 contains four subgroups of order 3: $G_1 = \{(1), (123), (132)\}$, $G_2 = \{(1), (124), (142)\}$, $G_3 = \{(1), (134), (143)\}$, $G_4 = \{(1), (134), (243)\}$.
  (a) Consider the submodule formally spanned by $\{G_1, G_2, G_3, G_4\}$ with the action $\sigma \cdot G_i = \{\sigma g \sigma^{-1} : g \in G_i\}$ for $\sigma \in A_4$. Give the decomposition of the module into irreducible submodules.
  (b) Now consider the submodule formally spanned by the left cosets of G_1, $\{\sigma G_1 : \sigma \in A_4\}$. Assume the action $\tau * \sigma G_1 = (\tau\sigma)G_1$ for $\tau \in A_4$. This is also a 4 dimensional module. Give the decomposition of the module into irreducible submodules.

(4) Let D_n be the dihedral group containing $2n$ elements be generated by the group elements x, y satisfying the relations $x^n = y^2 = e$ and $yx = x^{-1}y$. The elements of this group are

$$\{e, x, x^2, \cdots, x^{n-1}, y, yx, yx^2, \ldots, yx^{n-1}\} .$$

  (a) Show that for n odd the conjugacy classes are given by the subsets

$$\{e\} \cup \{y, yx, yx^2, \cdots, yx^{n-1}\} \cup \bigcup_{i=1}^{(n-1)/2} \{x^i, x^{-i}\}$$

  (b) Show that there are only two 1-dimensional representations of D_n for n odd by assuming that they are group homomorphisms and that χ satisfies the character relations and then determine the possible values of $\chi(x)$ and $\chi(y)$ for each irreducible χ.
  (c) Determine the number and dimensions of the irreducible characters of which are not 1-dimensional.
  (d) Find the character table for D_5.

(5) Let D_n act on $\mathcal{L}\{a_1, a_2\}$ by $x(a_1) = \zeta a_1$ where $\zeta = e^{2\pi i/n}$ and $y(a_1) = a_2$.
  (a) Determine $x(a_2)$ and $y(a_2)$ that makes this action a D_n module.
  (b) Let $V = \mathcal{L}\{a_1, a_2\}$ the D_5 module. Determine the character of V.
  (c) Determine how V decomposes into irreducible submodules.
  (d) Let $V^2 = \mathcal{L}\{a_1^2, a_2^2, a_1a_2\}$ be the D_5 submodule of polynomials of degree 2 in the basis elements of V. Determine the action of D_5 on the basis and tell how it decomposes into irreducible submodules.
  (e) Let $V^{\otimes 2} = \mathcal{L}\{a_1 \otimes a_1, a_1 \otimes a_2, a_2 \otimes a_1, a_2 \otimes a_2\}$ be the D_5 submodule of spanned by the tensors of degree 2 in the basis elements of V. Determine the action of D_5 on the basis and tell how it decomposes into irreducible submodules.