PROBLEMS FOR THE FINAL FOR MATH 6161

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For the final exam I want you to write programs which compute the character of a certain module in Sage. This requires several steps, none of which are complicated mathematically, but will require some effort to overcome the challenges of learning a computer language.

The modules that I would like you to tackle are subrings of either the polynomials $\mathbb{Q}[x_1, x_2, \cdots, x_n]$ or non-commutative polynomials $\mathbb{Q}\langle x_1, x_2, \cdots, x_n\rangle$.

1. Some submodules of the polynomial ring

The symmetric group S_n acts on monomials for $\sigma \in S_n$,

$$\sigma(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdots x_{\sigma(n)}^{a_n} \tag{1}$$

For a list of integers $(a_1, a_2, \cdots, a_r)$, define

$$\Delta_{(a_1,a_2,\cdots,a_r)} = \prod_{1 \le i < j \le r} (x_{a_i} - x_{a_j}) \tag{2}$$

In particular, if $r = 1$, then $\Delta_{(a)} = 1$ and if $a_i = a_j$ for some $i \neq j$ $\Delta_{(a_1,a_2,\cdots,a_r)} = 0$. Now for a filling of a diagram of a partition λ, with the entries given by T_{ij} for $1 \le i \le \ell(\lambda)$ and $1 \le j \le \lambda_i$, set

$$\Delta_T = \prod_{i=1}^{\ell(\lambda)} \Delta_{(T_{1i}, T_{2i}, \ldots, T_{\lambda'_i, i})} \tag{3}$$

Example 1. *If T is the standard tableau*

$$\begin{array}{|c|c|c|} \cline{1-2} 3 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array} \tag{4}$$

then

$$\Delta_T = (x_1 - x_3)(x_2 - x_5)$$

It is interesting to note that

Theorem 2.

$$M^\lambda = \mathcal{L}\{\Delta_T : T \textit{ standard of shape } \lambda\} \tag{5}$$

is an irreducible module which is isomorphic to Young's representation indexed by the same partition.

The modules that I want you to work on for your final project are based on these basis elements, but are not clear how they decompose into irreducibles. The best way of understanding how these modules decompose on general is to compute examples. I want to to write programs that allow you to collect data and make conjectures.

For a tableau T, let $m_i(T)$ be the number of labels of i in T. Define the content of a tableau to be the vector $content(T) = sort(m_1(T), m_2(T), \ldots, m_\ell(T))$ where the *sort* indicates that the entries of the tuple are sorted in weakly decreasing order (a partition).

Problem 1. *Let λ and μ be partitions of n. Let*

$$M^{(1)}_{\lambda,\mu} = \mathcal{L}\{\Delta_T : T \textit{ is column strict of shape } \lambda \textit{ and } content(T) = \mu\}\ . \tag{6}$$

Let $\ell = \ell(\mu)$, then $M^{(1)}_{\lambda,\mu}$ is an S_ℓ module.

Example 3. *Let $\lambda = (3,2)$ and $\mu = (2,2,1)$. There are 6 tableaux of shape $(3,2)$ and content $(2,2,1)$.*

$$\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\atop\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}\atop\begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 3&3\\\hline\end{array}\atop\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\atop\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\atop\begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 3&3\\\hline\end{array}\atop\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \tag{7}$$

Therefore $M^{(1)}_{\lambda,\mu}$ is spanned by the polynomials

$$\{(x_1-x_2)(x_1-x_3), (x_1-x_2)^2, (x_1-x_3)(x_2-x_3),$$
$$(x_1-x_2)(x_2-x_3), (x_1-x_2)(x_1-x_3), (x_1-x_3)^3\}$$

Notice that this list of 6 polynomials is not a basis since two of them are repeated.

Problem 2. *Let $\lambda^{(1)}$ and $\lambda^{(2)}$ be partitions of n. Define*

$$M^{(2)}_{\lambda^{(1)},\lambda^{(2)}} = \mathcal{L}\{\Delta_{T_1}\Delta_{T_2} : T_1 \textit{ and } T_2 \textit{ are standard of shape } \lambda^{(1)}, \lambda^{(2)} \textit{ respectively}\}\ . \tag{8}$$

Let $M^{(2)}_{\lambda^{(1)},\lambda^{(2)}}$ is an S_n module.

Example 4. *If $\lambda = (3)$, then there is only one standard tableau of that shape and $\Delta_T = 1$, and if $\lambda = (2,1)$ there are two standard tableaux* $\begin{array}{|c|}\hline 3\\\hline\end{array}\atop\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$ *and* $\begin{array}{|c|}\hline 2\\\hline\end{array}\atop\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}$ *therefore*

$$M^{(2)}_{(21),(3)} = \mathcal{L}\{(x_1-x_2), (x_1-x_3)\} \tag{9}$$

Problem 3. *Let $n > 1$ and define*

$$M^{(3)}_n = \mathcal{L}\{\partial^{a_1}_{x_1}\partial^{a_2}_{x_2}\cdots\partial^{a_n}_{x_n}\Delta_n : a_i \geq 0\} \tag{10}$$

where $\Delta_n = \prod_{1\leq i<j\leq n}(x_i - x_j)$.

An interesting generalization to this problem is where for a partition λ of n, replace Δ_n with Δ_T for all T standard of shape λ.

Example 5. *Note* $\Delta_2 = (x_1 - x_2)$ *so* $\partial_{x_1}\Delta_2 = 1$ *and* $\partial_{x_2}\Delta_2 = -1$ *so*

$$M_2^{(3)} = \mathcal{L}\{x_1 - x_2, 1\} \tag{11}$$

In every one of these problems you will need to complete the following steps.

- Find a spanning set for the module
- Find a basis for the module
- For any permutation, act on the basis and re-expand in terms of the basis elements
- to compute the character, sum over the coefficient of b_i in $\sigma(b_i)$ running over all basis elements b_i
- to compute the Frobenius image, compute

$$\frac{1}{n!}\sum_{\sigma\in S_n} character(\sigma) p_{cycle(\sigma)} \tag{12}$$

I think that the only steps you don't know how to do at this point involve Sage, namely how to find a basis for a module from a spanning set (linear algebra), and then how to compute with symmetric functions. In the meantime I will develop example programs of my own to show you how these steps can be overcome.

Everybody in the class will have a different module to work on. You see that they are similar in the types of programs you will have to write. I will give you two other modules which are submodules of the non-commutative polynomials on July 7.

The other type sof modules that I would like you to consider are those that are spanned by words (or non-commutative polynomials) rather than monomials in a commutative polynomial ring.

Note that there are two types of actions on words.

Let $w = w_1 w_2 \cdots w_k$ where $w_i \in \{1, 2, \ldots, n\}$. The left action (or the action on values) is defined as

$$\sigma(w) = \sigma(w_1)\sigma(w_2)\cdots\sigma(w_k) \tag{13}$$

Let $m_r(w) = \#\{w_i = r : 1 \le i \le \ell(w)\}$ and

$$W_\mu^n = \{w : \ell(w) = |\mu|, 1 \le w_i \le n, sort(m_1(w), m_2(w), \ldots, m_n(w)) = \mu\} \tag{14}$$

where *sort* indicates that the non-zero entries should be rearranged in weakly decreasing order. The set W_μ is the set of words $w = w_1 w_2 \cdots w_{|\mu|}$ of length $|\mu|$ whose vector of numbers of values appearing in w is μ.

Problem 4. *Let μ be a partition of length at most n and let λ be a partition of n. Define*

$$M^{(4)}_{\lambda,\mu} = \{E^{\lambda}_i(w) : w \in W^{|\lambda|}_{\mu}, 1 \leq i \leq f_\lambda\} \tag{15}$$

where the symmetric group $S_{|\mu|}$ acts on the left of these elements by permuting the values.

Example 6. *An example is $\lambda = (2,1)$ and $\mu = (1)$. Then using Sage we calculate (using the functions we wrote* `E_tab(Tab([2,1],i))`*) that*

$$E^{(21)}_1 = () - (12) - (132) + (13), E^{(21)}_2 = () + (12) - (123) - (13)$$

and $W^{(3)}_{(1)} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$.

$$\begin{aligned}
E^{(21)}_1(\mathbf{1}) &= \mathbf{1} - \mathbf{2} - \mathbf{3} + \mathbf{3}\\
E^{(21)}_2(\mathbf{1}) &= \mathbf{1} + \mathbf{2} - \mathbf{2} - \mathbf{3}\\
E^{(21)}_1(\mathbf{2}) &= \mathbf{2} - \mathbf{1} - \mathbf{1} + \mathbf{2}\\
E^{(21)}_2(\mathbf{2}) &= \mathbf{2} + \mathbf{1} - \mathbf{3} - \mathbf{2}\\
E^{(21)}_1(\mathbf{3}) &= \mathbf{3} - \mathbf{3} - \mathbf{2} + \mathbf{1}\\
E^{(21)}_2(\mathbf{3}) &= \mathbf{3} + \mathbf{3} - \mathbf{1} - \mathbf{1}
\end{aligned}$$

Hence $M^{(4)}_{(2,1),(1)} = \mathcal{L}\{\mathbf{1} - \mathbf{2}, \mathbf{1} - \mathbf{3}\}$

There is a second action on words. For $\sigma \in S_n$ and and $w = w_1 w_2 \cdots w_n$, then

$$(w)\sigma = w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n)} \ . \tag{16}$$

In this case we say that σ acts on the positions of w.

Problem 5. *Let μ and λ be partitions of n. Define*

$$M^{(5)}_{\lambda,\mu} = \{(w)E^{\lambda}_i : w \in W^{(|\mu|)}_{\mu}, 1 \leq i \leq f_\lambda\} \tag{17}$$

where the symmetric group $S_{|\mu|}$ acts on the left of these elements by permuting the values.

Example 7. *Lets start with a trivial example, $\mu = (3)$ and $\lambda = (2,1)$, then $W^3_{(3)} = \{\mathbf{111}, \mathbf{222}, \mathbf{333}\}$. But we have here that $(w)E_1 = (w)E_2 = 0$ for each w. Therefore $M^{(5)}_{(21),(3)} = \{0\}$.*

Example 8. *Because $E^{(3)}_1 = ()$, $M^{(5)}_{(3),(21)}$ is equal to the linear span of the elements of $W^3_{(21)} = \{\mathbf{112}, \mathbf{121}, \mathbf{211}, \mathbf{113}, \mathbf{131}, \mathbf{311}, \mathbf{223}, \mathbf{232}, \mathbf{322}, \mathbf{221}, \mathbf{212}, \mathbf{122}, \mathbf{331}, \mathbf{313}, \mathbf{133}, \mathbf{332}, \mathbf{323}, \mathbf{233}\}$.*

To compute $M^{(5)}_{(21),(21)}$ there are 36 calculations to complete since we again have

$$E^{(21)}_1 = () - (12) - (132) + (13), E^{(21)}_2 = () + (12) - (123) - (13)$$

and there are 9 words in $W^3_{(21)}$.

$$(\mathbf{112})E^{(21)}_1 = \mathbf{112} - \mathbf{112} - \mathbf{211} + \mathbf{211} = 0$$

$$(\mathbf{112})E^{(21)}_2 = \mathbf{112} + \mathbf{112} - \mathbf{121} - \mathbf{211}$$

$$(\mathbf{121})E_1^{(21)} = \mathbf{121} - \mathbf{211} - \mathbf{112} + \mathbf{121}$$
$$(\mathbf{121})E_2^{(21)} = \mathbf{121} + \mathbf{211} - \mathbf{211} - \mathbf{121} = 0$$
$$(\mathbf{211})E_1^{(21)} = \mathbf{211} - \mathbf{121} - \mathbf{121} + \mathbf{112}$$
$$(\mathbf{211})E_2^{(21)} = \mathbf{211} + \mathbf{121} - \mathbf{112} - \mathbf{112}$$

This calculation is repeated then 6 *times (5 more times) with* $\sigma((w)E_i^{(21)}) = (\sigma(w))E_i^{(21)}$ *for each* $\sigma \in S_3$. *These* 6 *elements span a space of dimension two spanned by* $\{\mathbf{112} - \mathbf{121}, \mathbf{112} - \mathbf{211}\}$ *and so the total dimension of* $M^{(5)}_{(21),(21)}$ *is* 12.