MATH 6161 - Algebraic Combinatorics: Symmetric Functions

Summer 2014

Definition: A *permutation* is bijection $\sigma : [n] \to [n]$. There several notations that can be used to represent a permutation. Several instances of them are:

- One line notation: $\sigma(1)\sigma(2)\cdots\sigma(n)$
- Two line notation:
$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$
- Cycle notation: $(\sigma(i_1), \sigma(i_2), \cdots, \sigma(i_r)) \cdots (\sigma(i_m), \sigma(i_{m+1}), \cdots, \sigma(i_n))$

Let the group of permutations on n letters be denoted by $\mathfrak{S}_n$. We say that $\sigma \in \mathfrak{S}_n$ is a permutation with **cycle structure** $(m_1, m_2, \ldots, m_n)$ if σ has precisely m_1 one-cycles, m_2 two-cycles and so on. Note that this implies $1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n$.

Proposition: If $\pi = (i_1, i_2, \cdots, i_r) \cdots (i_m, i_{m+1}, \cdots, i_n)$ in cycle notation, then for any $\sigma \in \mathfrak{S}_n$,
$$\sigma\pi\sigma^{-1} = (\sigma(i_1), \sigma(i_2), \cdots, \sigma(i_r)) \cdots (\sigma(i_m), \sigma(i_{m+1}), \cdots, \sigma(i_n)).$$

Definition: Two permutations σ and π have the same cycle structure if and only if they are conjugate. That is,
$$\pi = \alpha\sigma\alpha^{-1}$$
for some $\alpha \in \mathfrak{S}_n$.

Proposition: Let $\pi \in \mathfrak{S}_n$ be a permutation with cycle structure $(m_1, m_2, \ldots, m_n)$. Then of permutations in $\mathfrak{S}_n$ with the same cycle structure as π is
$$\frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!}.$$

Proof: As π and σ have the same cycle type if and only if they are in the same conjugacy class os $\mathfrak{S}_n$, it suffices to prove that the number of permutations σ that fix π under the action of conjugation is
$$1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!.$$

In order for a permutation σ to fix π, its action restricted to the m_i cycles of length i must be *at least* one of the following:

- Permute the m_i cycles of length i amongst themselves in $m_i!$ ways (for instance, $(134)(256) = (256)(134)$)
- Pick the first element of every cycle of length i in i^{m_i} ways (for instance, $(134) = (341) = (413)$).

Thus, since both of these are precisely the ways in which a permutation can fix π, we see that the number of permutations with the cycle structure of π is by the orbit stabilizer theorem
$$\frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!}.$$

■

Definition: Let G be a finite group, let $GL_n(\mathbb{C})$ be the set of invertible $n \times n$ matrices. Then a representation of G is a homomorphism $\psi : G \to GL_n(\mathbb{C})$, which we say has degree n.

Example: Let $\mathfrak{S}_n$ act on $\{\underline{1}, \underline{2}, \ldots, \underline{n}\}$ in the natural way. That is, $\sigma\underline{i} = \underline{\sigma(i)}$ for all $i \in [n]$.

For instance, when $n = 3$,
$$\begin{aligned} (1)(2)(3)\underline{1} &= \underline{1} \\ (123)\underline{2} &= \underline{3} \\ (12)(3)\underline{3} &= \underline{3} \end{aligned}$$

Now, we compute the matrix of the permutations of $\mathfrak{S}_3$ in the standard basis:

$$X((1)(2)(3)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad X((13)(2)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$X((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad X((12)(3)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X((1)(23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

□

Example: Let $V := \mathcal{L}\{v_1, v_2\}, v_3 := -v_1 - v_2$. We let $\mathfrak{S}_3$ act on the basis elements in the natural way, that is

$$\sigma v_i = v_{\sigma(i)}.$$

Now, we compute the action of group elements on the basis to find their representations

$$\begin{aligned} (12)v_1 &= v_2 & (12)v_2 &= v_1 \\ (13)v_1 &= -v_1 - v_2 & (13)v_2 &= v_2 \\ (23)v_1 &= v_1 & (23)v_2 &= -v_1 - v_2 \\ (123)v_1 &= v_2 & (123)v_2 &= -v_1 - v_2 \\ (132)v_1 &= -v_1 - v_2 & (132)v_2 &= v_1 \\ (1)v_1 &= v_1 & (1)v_2 &= v_2 \end{aligned}$$

And so, the representation of our group in this $\mathfrak{S}_3$**-module** is

$$X((12) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad X((13)) = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$X((23)) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \qquad X((123)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$X((132)) = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \qquad X((1)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Definition: Let G be a finite group. We say that vector space V is a $G-$**module** if there is a group homomorphism $\psi : G \to GL(V)$. That is, we would like ψ to satisfy the following properties

1. $g\mathbf{v} \in V$
2. $g(c\mathbf{v} + d\mathbf{w}) = c(g\mathbf{v}) + d(g\mathbf{w})$
3. $(gh)\mathbf{v} = g(h\mathbf{v})$
4. $id\mathbf{v} = v$

for all $g, h \in G$, $\mathbf{v}, \mathbf{w} \in V$ and scalars $c, d \in \mathbb{C}$.

Example: An *inversion* of a permutation σ is a pair (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$. Let $inv(\sigma)$ denote the number of inversions in σ. Then the mapping $sgn : \sigma \mapsto (-1)^{inv(\sigma)}$ is a homomorphism and is known as the sign representation of $\mathfrak{S}_n$.

Example: The mapping which maps all of G to the identity of a vector space is known as the *trivial representation* of G.

For instance, if $V = \mathcal{L}\{w\}$, then for any group G,

$$g.w = w$$

for all $g \in G$.
Definition: A subspace W of a $G-$module V is called a **submodule** of V if W is $G-$invariant. This means that $g.w \in W$ for all $g, \in G$ and $w \in W$.
Example: (Trivial Submodule) Every $G-$module V has two trivial submodules $\{0\}$ and V itself.
Remark: Let $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ and $\mathcal{C} = \{w_1, w_2, \cdots, w_n\}$ be bases for a $G-$module V. Then there are coefficients,

$$g(v_i) = \sum_{j=1}^{n} a_{ji} v_j$$

$$g(w_i) = \sum_{j=1}^{n} b_{ji} w_j$$

and

$$v_i = \sum_{j=1}^{n} t_{ji} w_j.$$

If we first compute the action of g on v_i and expand in the basis $\mathcal{C}$, then we have

$$g(v_i) = \sum_{j=1}^{n} a_{ji} v_j = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ji} t_{kj} w_k$$

If instead we compute the action of g on v_i by expanding in the basis $\mathcal{C}$ followed by the action of g on the $\mathcal{C}$ basis we have

$$g(v_i) = \sum_{j=1}^{n} t_{ji} g(w_j) = \sum_{j=1}^{n} \sum_{k=1}^{n} t_{ji} b_{kj} w_k$$

Now let $T = [t_{ji}]_{1 \leq i,j \leq n}$ be the matrix of coefficients for the change of basis matrix between the $\mathcal{B}$ basis and $\mathcal{C}$ basis.

The coefficient of w_k in $g(v_i)$ using the first of these two equations is equal to the (k, i) entry in the matrix $T \cdot X_{\mathcal{B}}(g)$ where $X_{\mathcal{B}}(g) = [a_{ji}]_{1 \leq i,j \leq n}$.

The coefficient of w_k in $g(v_i)$ using the second of these two equations is equal to the (k, i) entry in the matrix $X_{\mathcal{C}}(g) \cdot T$ where $X_{\mathcal{C}}(g) = [b_{ji}]_{1 \leq i,j \leq n}$.

Since these two quantities must be equal for the action to be consistent on the the bases, we must have

$$T \cdot X_{\mathcal{B}}(g) = X_{\mathcal{C}}(g) \cdot T$$

$\square$

Example: Let $V = \mathcal{L}\{\underline{1}, \underline{2}, \underline{3}\}$ and $G = \mathfrak{S}_3$ act on it in the natural way. We claim that $W = \mathcal{L}\{\underline{1} + \underline{2} + \underline{3}\}$ is a submodule of G:

$$\sigma(\underline{1} + \underline{2} + \underline{3}) = \sigma(\underline{1}) + \sigma(\underline{2}) + \sigma(\underline{3}) = \underline{1} + \underline{2} + \underline{3} \in W$$

as σ is a bijection for all $\sigma \in \mathfrak{S}_3$. As this holds for every basis element, it holds for all W.
Remark: Although V decomposes to the direct sum of $W = \mathcal{L}\{\underline{1} + \underline{2} + \underline{3}\}$ and $U = \mathcal{L}\{2, 3\}$ as vector spaces, U is not a submodule of V: $(13)(\underline{3}) = \underline{1} \notin W$, for instance.
However, we can find the unique submodule $W^{\perp}$ for which $V = W \bigoplus W^{\perp}$ as follows:
Define the inner product $<,>$ on V by

$$< \underline{i}, \underline{j} >= \delta_{\underline{i},\underline{j}}$$

for the basis elements $\underline{i}, \underline{j} \in \{\underline{1}, \underline{2}, \underline{3}\}$ and then we extend linearly in the first variable and conjugate linearly in the second. Now, we search for the orthogonal complement of W under this inner product:

$$W^{\perp} = \{a\underline{1} + b\underline{2} + c\underline{2} : a, b, c \in \mathbb{C} \text{ and } a + b + c = 0\}.$$

This is a submodule with basis $\{3-2, 3-1\}$. In conclusion,

$$\mathcal{L}\{\underline{1},\underline{2},\underline{3}\} = \mathcal{L}\{\underline{1}+\underline{2}+\underline{3}\} \oplus \mathcal{L}\{3-2, 3-1\}.$$

which is a decomposition of V into into its submodules. $\square$

Definition: A $G-$module V is *irreducible* if it has no nontrivial submodules.

Lemma: If W is a submodule of V and and $<,>$ is a $G-$invariant scalar product, then $W^\perp = \{v \in V :< v, w >= 0 \text{ for all } w \in W\}$ is also a submodule of V and $V = W \oplus W^\perp$.

Proof: Fix $v \in W^\perp$. Let $g \in G$ and $w \in W$. Then

$$\begin{aligned} < gv, u > &=< v, g^{-1}u > \qquad \text{by the } G\text{−invariance} \\ &= 0 \qquad \text{As } W \text{ is a submodule of } V \end{aligned}$$

which proves that $W^\perp$ is a submodule of V.

Finally, let $\mathcal{B} = \{\overbrace{\alpha_1, \alpha_2, \dots, \alpha_k}^{W}, \underbrace{\alpha_{k+1}, \dots, \alpha_n}_{W^\perp}\}$ be an orthonormal basis with respect to the inner product $<,>$ with the first k vectors being a basis for W and rest being a basis for $W^\perp$.

Fix $v \in V$ and let w to be the projection of v into W:

$$w :=< v, \alpha_1 > \alpha_1 + \cdots + < v, \alpha_k > \alpha_k$$

and then $v = w + (v - w) \in W \oplus W^\perp$. $\blacksquare$

Theorem: (Maschke's Theorem)

If G is a finite group and V is a $G-$module over $\mathbb{C}$, then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

where every V_i is an irreducible submodule of V.

Proof: Let $<,>$ be the inner product defined by $< i, j >= \delta_{i,j}$, where i, j are basis elements of V. If this inner product is not $G-$invariant, then we define $<,>'$ as follows in order to make use of the above lemma:

$$\begin{aligned} < u, v >' &:= \sum_{g \in G} < gu, gv > \\ \implies < hu, hv >' &= \sum_{g \in G} < ghu, ghv > \\ &=< u, v >' \end{aligned}$$

for all $u, v \in V, h \in G$. This implies that for a finite group G, every module has some $G-$invariant product defined on it.

Finally, to prove the theorem, we utilize induction on the dimension of a module and appeal to the results of the lemma above. $\blacksquare$

Example: If G is a finite group and G acts on itself, then $\mathcal{L}\{g \in G\}$ is a $G-$module. This is known as the *regular* representation of G.

For instance, when $G = C_4 =< g : g^4 = e >$ is the cyclic group of order 4, then the matrix representations in $V_1 = \mathcal{L}\{e, g, g^2, g^3\}$ of C_4 are

$$X_1(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad X_1(g) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$X_1(g^2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad X_1(g^3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $v_1 = e + g + g^2 + g^3$. Then $V_1 \subseteq \mathcal{L}\{v_1\}$ as a submodule.
Let $V_2 = \mathcal{L}\{e + g + g^2 + g^3, g, g^2, g^3\}$, then the matrix representations of G in V_2 are

$$X_2(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad X_2(g) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$X_2(g^2) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad X_2(g^3) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Let $V_3 = \mathcal{L}\{e - g, g - g^2, g^2 - g^3\}$. Then $V_3 = V_2^{\perp}$ and the representations of G in $V_1 \bigoplus V_3$ is

$$X_3(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad X_3(g) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$X_3(g^2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad X_3(g^3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Remark: Although we broke down the module V_1 into two submodules, not both are irreducible and V_1 can be decomposed further.
Definition: Let V and W be two $G-$modules. Then a linear map $\phi : V \to W$ is a *G-homomorphism* if

$$\phi(gv) = g\phi(v)$$

for all $g \in G, v \in V$.
Proposition: Let $\phi : V \to W$ be a $G-$homomorphism, then

1. $\phi(V)$ is a submodule of W.
2. $\text{Ker}(\phi)$ is a submodule of V.

Proof: Let $w \in \phi(V)$. Fix $g \in G$ and suppose $\phi(v) = w$. Then

$$gw = g\phi(v) = \phi(gv) \in \phi(V)$$

as V is a $G-$module.
Now, suppose $v \in \text{Ker}(\phi)$ and fix $g \in G$. Then

$$\phi(gv) = g\phi(v) = g(0) = 0$$

and so $gv \in \text{Ker}(\phi)$ if $v \in V$. ■
Remark: If there exists a $G-$homomorphism from V to W, then there exists a matrix T such that

$$TX_v(g) = X_W(g)T.$$

Theorem: (Schur's Lemma)
If V and W are irreducible representations with $X_V(g)T = TX_W(g)$ for all $g \in G$, then T is either zero or is invertible.
Proof: By the last proposition,

$$\begin{aligned} V \text{ irreducible} &\implies \text{Ker}(\phi) = \{0\} \text{ or } \text{Ker}(\phi) = V \\ W \text{ irreducible} &\implies \phi(V) \ = \{0\} \text{ or } \phi(V) = V \end{aligned}$$

and so the theorem follows from the remark. ■

Now, suppose V is an irreducible $G-$module . If T is a matrix such that

$$TX(g) = X(g)T$$

for all $g \in G$, then

$$TX = XT \implies (T - cI)X = X(T - cI) \qquad \text{as } cI \text{ commutes with } X \text{ for all } c \in \mathbb{C}$$

but as $\mathbb{C}$ is algebraically closed, we may pick c to be an eigenvalue of $\mathbb{C}$ so $T - cI$ satisfies the hypothesis of Schur's Lemma and it is not invertible, which implies $T - cI = 0$ and so $T = cI$ for some $c \in \mathbb{C}$. We summarize this as follows:

Corollary: If V is an irreducible G-module, then T commutes with the matrix representation of G if and only if $T = cI$ for some $c \in \mathbb{C}$. ■

Example: Consider the $\mathfrak{S}_2-$modules $V = \mathbb{C}[x_1, x_2] = \mathcal{L}\{1, x_1, x_2, x_1x_2, x_1^2, x_2^2, \ldots\}$ where

$$\begin{aligned}(1)(2)x_1^a x_2^b &= x_1^a x_2^b \\ (12)x_1^a x_2^b &= x_1^b x_2^a\end{aligned}$$

and so, the action of $\mathfrak{S}_2$ preserves the degree of the module.

Hence, the monomials of degree r form a submodule of of V and

$$\begin{aligned}V &= \mathcal{L}\{1\} \oplus \mathcal{L}\{x_1, x_2\} \oplus \mathcal{L}\{x_1^2, x_2^2, x_1x_2\} \oplus \cdots \\ &= \bigoplus_{r \geq 0} V_r\end{aligned}$$

where V_r denotes the monomials of degree r.

Let $R^{\mathfrak{S}_2}$ denote the **Reynolds operator** of $\mathfrak{S}_2$ which acts on V by $v \mapsto (1)(2)v + (12)v$. For instance,

$$\begin{aligned}R^{\mathfrak{S}_2}(1) &= 1 + 1 = 2 \\ R^{\mathfrak{S}_2}(x_1) &= x_1 + x_2 \\ R^{\mathfrak{S}_2}(x_2) &= x_1 + x_2 \\ &\vdots \\ R^{\mathfrak{S}_2}(x_1^a x_2^b) &= \begin{cases} x_1^a x_2^b + x_1^b x_2^a & a \neq b \\ 2x_1^a x_2^b & a = b \end{cases}\end{aligned}$$

Note that $\mathcal{L}\{x_1^a x_2^b + x_1^b x_2^a : 0 \leq a \leq b\}$ is a submodule of V.

We define an inner product on V by

$$< x_1^a x_2^b, x_1^c x_2^d >= \begin{cases} 1 & a = c \text{ and } b = d \\ 0 & else \end{cases}$$

and we find that $x_1^a x_2^b - x_1^b x_2^a$ is orthogonal to all of $\mathcal{L}\{x_1^c x_2^d + x_1^d x_2^c : 0 \leq c \leq d\}$:

$$< x_1^a x_2^b - x_1^b x_2^a, x_1^c x_2^d + x_1^d x_2^c > = \left\{\begin{matrix} 1 & a = c, b = d \\ 0 & else \end{matrix}\right\} + \left\{\begin{matrix} 1 & a = d, b = c \\ 0 & else \end{matrix}\right\} - \left\{\begin{matrix} 1 & b = c, a = d \\ 0 & else \end{matrix}\right\} - \left\{\begin{matrix} 1 & b = d, a = c \\ 0 & else \end{matrix}\right\}$$

and $\mathcal{L}\{x_1^a x_2^b - x_1^b x_2^a\}$ is a submodule as

$$(12)(x_1^a x_2^b - x_1^b x_2^a) = x_1^b x_2^a - x_1^a x_2^b = -(x_1^a x_2^b - x_1^b x_2^a)$$

Therefore, $\mathcal{L}\{x_1^a x_2^b - x_1^b x_2^a\}$ is a submodule for all $a < b$. We see that our V_r can be broken down as follows

$$
\begin{aligned}
V_0 &= \mathcal{L}\{1\} \\
V_1 &= \mathcal{L}\{x_1, x_2\} = \mathcal{L}\{x_1 + x_2\} \oplus \mathcal{L}\{x_1 - x_2\} \\
V_2 &= \mathcal{L}\{x_1^2, x_2^2, x_1x_2\} = \mathcal{L}\{x_1^2 + x_2^2\} \oplus \mathcal{L}\{x_1x_2\} \oplus \mathcal{L}\{x_1^2 - x_2^2\}
\end{aligned}
$$

and in general we see that

$$
V_r = \mathcal{L}\overbrace{\{x_1^r, x_1^{r-1}x_2, x_1^{r-2}x_2^2, \ldots, x_1^2x_2^{r-1}, x_2^r\}}^{r+1 \text{ basis elements}} \supseteq \left[\bigoplus_{\substack{a+b=r \\ a\leq b}} \mathcal{L}\{x_1^a x_2^b + x_1^b x_2^a\}\right] \oplus \left[\bigoplus_{\substack{a+b=r \\ a<b}} \mathcal{L}\{x_1^a x_2^b - x_1^b x_2^a\}\right]
$$

with equality as on the right hand side we have that

$$
\begin{aligned}
dim\left(\left[\bigoplus_{\substack{a+b=r \\ a\leq b}} \mathcal{L}\{x_1^a x_2^b + x_1^b x_2^a\}\right] \oplus \left[\bigoplus_{\substack{a+b=r \\ a<b}} \mathcal{L}\{x_1^a x_2^b - x_1^b x_2^a\}\right]\right) &= dim\left[\bigoplus_{\substack{a+b=r \\ a\leq b}} \mathcal{L}\{x_1^a x_2^b + x_1^b x_2^a\}\right] \\
&\quad + dim\left[\bigoplus_{\substack{a+b=r \\ a<b}} \mathcal{L}\{x_1^a x_2^b - x_1^b x_2^a\}\right] \\
&= \left\lceil \frac{r+1}{2} \right\rceil + \left\lfloor \frac{r+1}{2} \right\rfloor \\
&= r+1
\end{aligned}
$$

and so equality holds.
Next, we define the generating function $dim_q(V)$[1] by

$$
dim_q(V) = \sum_{n\geq 0} dim(V_n) q^n
$$

where V_n is the space of spanned by homogeneous elements of V of degree n.
In our example,

$$
\begin{aligned}
dim_q(\mathbb{C}[x_1, x_2]) &= 1 + 2q + 3q^2 + 4q^3 + \cdots + (r+1)q^r + \cdots \\
&= D_q\left(\frac{1}{1-q}\right) = \frac{1}{(1-q)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
dim_q\left(\bigoplus_{r\geq 0}\left(\bigoplus_{\substack{a+b=r \\ a\leq b}} \mathcal{L}\{x_1^a x_2^b + x_1^b x_2^a\}\right)\right) &= 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + 4q^7 + \cdots \\
&= (1+q)\left[1 + 2q^2 + 3q^4 + 4q^6 + \cdots\right] \\
&= (1+q)\left[\sum_{r\geq 0}(r+1)q^{2r}\right] \\
&= (1+q)\left[\frac{1}{(1-q^2)^2}\right]
\end{aligned}
$$

[1]This is known as the Hilbert Series of V.

as $\dfrac{1}{(1-q)^2}$ is the ordinary generating function of $\{r+1\}_{r\geq 0}$ and hence

$$dim_q\left(\bigoplus_{r\geq 0}\left(\bigoplus_{\substack{a+b=r\\ a\leq b}}\mathcal{L}\{x_1^a x_2^b + x_1^b x_2^a\}\right)\right) = (1+q)\left[\frac{1}{(1-q^2)^2}\right]$$
$$= \frac{1}{(1-q)(1-q^2)}$$

and hence

$$dim_q\left(\bigoplus_{r\geq 0}\left(\bigoplus_{\substack{a+b=r\\ a<b}}\mathcal{L}\{x_1^a x_2^b - x_1^b x_2^a\}\right)\right) = \frac{1}{(1-q)^2} - \frac{1}{(1-q)(1-q^2)}$$
$$= \frac{q}{(1-q)(1-q^2)}$$

$\square$

Now, we create new $G-$modules from old ones:
Given $V \supseteq U$ and W $G-$modules with bases $\mathcal{B}$ and $\mathcal{C}$ and representations X and Y, respectively, and if dim $V = n$, dim $U = d$ and dim $W = m$, then

Module	**Notation**	**Dimension**	**Representation**
Direct Sum Module	$V \oplus W$	$n+m$	$\left[\begin{array}{c\|c} X(g) & 0 \\ \hline 0 & Y(g)\end{array}\right]$
Tensor Product Module	$V \otimes W$	$n \cdot m$	$\begin{bmatrix} a_{11}Y(g) & a_{12}Y(g) & \cdots & a_{1n}Y(g) \\ a_{21}Y(g) & a_{22}Y(g) & \cdots & a_{2n}Y(g) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}Y(g) & a_{n2}Y(g) & \cdots & a_{nn}Y(g)\end{bmatrix}$
Quotient Module	V/U	$n-d$	$X(g)\|_{\mathcal{R}}$

where $\mathcal{R}$ is the subset of $\mathcal{B}$ that excludes the basis elements of U. The direct sum and tensor product modules have the the bases $\mathcal{B}\cup\mathcal{C}$, $\{v_1\otimes w_1, \ldots, v_1\otimes w_m, \ldots, v_n\otimes w_1, \ldots, v_n\otimes w_m\}$ while the quotient module has a spanning set $\{U+v : v \notin U\}$. $\square$

Exercise: Let $\mathbb{C}[x_1,x_2,x_3]^{\mathfrak{S}_3}$ denote the space of polynomials in 3 variables over $\mathbb{C}$ which are invariant under the action of $\mathfrak{S}_3$ of

$$\sigma(f(x_1,x_2,x_3)) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

Find $dim_q(\mathbb{C}[x_1,x_2,x_3]^{\mathfrak{S}_3})$.

Step 1: Conjecture the answer.
Method 1: figure out the graded dimensions for the first few terms and then look the answer up in the "online integer sequence database."
Method 2: recall that a basis for the submodule $\mathbb{C}[x_1,x_2]^{\mathfrak{S}_2}$ = submodule consisting of those elements invariant under the $\mathfrak{S}_2$ action is

$$\{x_1^a x_2^b + x_1^b x_2^a \text{ for } a \geq b\}\ .$$

Use this to guess (big leap here) that a basis for $\mathbb{C}[x_1,x_2,x_3]^{\mathfrak{S}_3}$ is the set

$$\{R^{\mathfrak{S}_3}(x_1^a x_2^b x_3^c) \text{ for } a \geq b \geq c \geq 0\}$$

Step 2: Show that the set above is a basis by demonstrating that it spans and is linear independent.

Step 3: Show that generating function for the number of elements in the set

$$\{(a,b,c) : a \geq b \geq c \geq 0\}$$

is equal to $\left(\frac{1}{(1-q)(1-q^2)}\right)\frac{1}{(1-q^3)}$.
Conclude that

$$dim_q(\mathbb{C}[x_1,x_2,x_3]^{\mathfrak{S}_3}) = \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

$\square$

Recall that a matrix T commutes with an irreducible representation X of a if and only if $T = cI$ for some constant $c \in \mathbb{C}$. Now, suppose that

$$X = \begin{bmatrix} X_1(g) & 0 \\ 0 & X_2(g) \end{bmatrix}$$

for some X_1 and X_2 irreducibles. Then for any matrix T such that

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

and

$$X(g)T = \begin{bmatrix} X_1(g)T_{11} & X_1(g)T_{12} \\ X_2(g)T_{21} & X_2(g)T_{22} \end{bmatrix} = \begin{bmatrix} X_1T_{11} & X_2(g)T_{12} \\ X_1T_{21} & X_2(g)T_{22} \end{bmatrix} = TX(g)$$
$$\implies T_{11} = c_1I \text{ and } T_{22} = c_2I$$

and

$$\begin{Bmatrix} T_{12} = 0 = T_{21} & \text{if } X_1 \ncong X_2 \\ T_{12} = c_1'I \text{ and } T_{21} = c_2'I & X_1 \cong X_2 \end{Bmatrix}$$

To summarize if $X = 2X_1$, then

$$T = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \otimes I$$

for some $c_{11}, c_{12}, c_{21}, c_{22} \in \mathbb{C}$ and otherwise

$$T = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix} \otimes I$$

for some $c_{11}, c_{22} \in \mathbb{C}$.
More generally, if $X = m_1X_1 \otimes m_2X_2$, then if T commutes with X, we find that

$$T = \left[\begin{array}{cccc|cccc} c_{11}I & c_{12}I & \cdots & c_{1m_1}I & & & & \\ c_{21}I & c_{22}I & \cdots & c_{2m_1}I & & & & \\ \vdots & & \ddots & \vdots & & & & \\ c_{m_11}I & c_{m_12}I & \cdots & c_{m_1m_1}I & & & & \\ \hline & & & & a_{11}I & a_{12}I & \cdots & a_{1m_2}I \\ & & & & a_{21}I & a_{22}I & \cdots & a_{2m_2}I \\ & & & & \vdots & & \ddots & \vdots \\ & & & & a_{m_21}I & a_{m_22}I & \cdots & a_{m_2m_2}I \end{array}\right]$$
$$= \left[\begin{array}{c|c} C \otimes I_{d_1} & 0 \\ \hline 0 & A \otimes I_{d_2} \end{array}\right]$$

and if $Com\ X = \{T : TX(g) = X(g)T, \forall g \in G\}$, then

$$dim(Com(X)) = m_1^2 + m_2^2$$

and $degX = m_1 d_1 + m_2 d_2$ if $deg(X_1) = d_1$ and $deg(X_2) = d_2$.
We summarize and generalize even further:

Proposition: If $X = \bigotimes_{i=1}^{k} m_i X_i$, then

$$dim(Com(X)) = \sum_{i=1}^{k} m_i^2$$
$$deg(X) = \sum_{i=1}^{k} m_i d_i$$

where $d_i = deg(X_i)$. $\blacksquare$

Definition: Let X be a representation. Then associated with X is χ, the *character* of a representation, which is the mapping $X(g) \mapsto tr(X(g))$.

Example: For instance, if

$$X = \left[\begin{array}{c|c} X_1 & 0 \\ \hline 0 & X_2 \end{array}\right]$$

then $\chi(g) = \chi^{(1)}(g) + \chi^{(2)}(g)$.

To prove that χ is well defined, it is sufficient to note that the trace is invariant under conjugation. In fact, this also proves that the trace is a *class function* meaning that it is constant on the conjugacy classes of G.

Fact: if G acts by permuting the basis elements, then

$$\chi(g) = |fix(g)|.$$

Define an inner product $<,>$ on characters by

$$\begin{aligned} <\chi, \psi> &:= \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} \end{aligned}$$

as we may pick a unitary basis.

Fact: If X and Y are irreducible as representations with characters χ and ψ, then

$$<\chi, \psi> = \delta_{\chi,\psi}.$$

Consequences: χ is irreducible $\iff <\chi, \chi> = 1$

Proof: If χ is irreducible, then $<\chi, \chi> = 1$. Conversely, if it is not irreducible, then $\chi = \chi^{(1)} + \chi^{(2)}$, for some other characters $\chi^{(1)}, \chi^{(1)}$ and so

$$<\chi, \chi> \neq 1.$$

$\square$

Furthermore, if $X = \bigotimes_{i=1}^{k} m_i X^{(i)}$ where all $X^{(i)}$ are irreducible, then

$$<\chi^X, \chi^{(i)}> = <m_1\chi^{(1)} + m_2\chi^{(2)} + \cdots + m_k\chi^{(k)}, \chi^{(i)}> = m_i$$

and

$$<\chi^X, \chi^X> = \sum_{i,j=1}^{k} <m_i\chi^{(i)}, m_j\chi^{(j)}> = m_1^2 + m_2^2 + \cdots + m_k^2.$$

Question: Given a group G and a matrix representation X, how do we decompose X?

Step 1: List all irreducible characters of G. Say, $\{\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(k)}\}$.

Step 2: Compute $< \chi^X, \chi^{(i)} >= m_i$.

How to do Step 1: Notice that $\chi^{(i)}$ is constant on the conjugacy classes of $G = \dot{\bigcup} C_i$. Now, define

$$\kappa_{C_i}(g) = \begin{cases} 1 & \text{if } g \in C_i \\ 0 & \text{else} \end{cases}$$

and note that these form a basis for the vector space of all class functions on G:

$$\chi(g) = \chi(g_1)\kappa_{C_1}(g) + \chi(g_2)\kappa_{C_2}(g) + \cdots + \chi(g_d)\kappa_{C_d}(g),$$

where $g_i \in C_i$.

Example: If $G = \mathfrak{S}_3$, its conjugacy classes are

$$\begin{aligned} C_1 &= \{(1)\} \\ C_2 &= \{(12), (13), (23)\} \\ C_3 &= \{(123), (132)\} \end{aligned}$$

and our basis consists of

$$\kappa_1(g) = \begin{cases} 1 & g = e \\ 0 & \text{else} \end{cases}$$

$$\kappa_2(g) = \begin{cases} 1 & g \in C_2 \\ 0 & \text{else} \end{cases}$$

$$\kappa_3(g) = \begin{cases} 1 & g \in C_3 \\ 0 & \text{else} \end{cases}$$

If $V = \mathcal{L}\{\underline{1}, \underline{2}, \underline{3}\}$ then

$$\begin{aligned} \chi^V((1)) &= 3 \\ \chi^V((12)) &= 1 \\ \chi^V((123)) &= 0 \end{aligned}$$

and so $\chi^V(g) = 3\kappa_1(g) + \kappa_2(g)$.
Let $V_1 = \mathcal{L}\{\underline{1} + \underline{2} + \underline{3}\}$. Then $\chi^{(1)} = 1\kappa_1 + 1\kappa_2 + 1\kappa_3$.
Let X_2 be the sign representation. Then $\chi^{(2)} = \kappa_1 - \kappa_2 + \kappa_3$.
Now, since

$$\begin{aligned} < \chi^V, \chi^{(1)} > &= \frac{1}{6}(3 + 3 + 0) = 1 \\ < \chi^V, \chi^{(2)} > &= \frac{1}{6}(3 + 3(-1) + 0) = 0 \end{aligned}$$

we know that $V \cong V_1 \oplus \mathcal{L}\{\underline{1} - \underline{2}, \underline{2} - \underline{3}\}$. Let $V_2 = \mathcal{L}\{\underline{1} - \underline{2}, \underline{2} - \underline{3}\}$. Then we see that

$$\begin{aligned} \chi^{V_2}((1)) &= 2 \\ \chi^{V_2}((12)) &= 0 \\ \chi^{V_2}((132)) &= -1 \end{aligned}$$

and so $\chi^{V_2} = 2\kappa_1 + \kappa_3$. Moreover,

$$< \chi^{(1)}, \chi^{V_2} > = 0$$
$$< \chi^{(2)}, \chi^{V_2} > = 0$$
$$< \chi^{V_2}, \chi^{V_2} > = 1$$

and so χ^{V_2} is irreducible. Therefore, the **character table** of $\mathfrak{S}_3$ is

	C_1	C_2	C_3
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)} = \chi^{V_2}$	2	0	-1

□

In general:
Proposition: Let $V = \mathcal{L}\{g \in G\}$ be the group algebra of G. Then

$$\chi^V(g) = \begin{cases} |G| & g = e \\ 0 & else \end{cases}$$

Proof: Let $\chi^{(i)}$ be an irreducible character. Note that $\chi^{(i)}(e) = dim(X^i)$. So,

$$\begin{aligned} < \chi^V, \chi^V > &= \frac{1}{|G|}\sum_{g\in G}\chi^V(g)\chi^{(i)}(g) \\ &= \frac{1}{|G|}(\chi^V(e)\chi^{(i)}(e) \qquad \text{as } \chi^V(g) = 0 \text{ for } g \neq e \\ &= dim\chi^{(i)} = m_i \end{aligned}$$

and

$$\begin{aligned} < \chi^V, \chi^V > &= \frac{1}{|G|}\sum_{g\in G}\chi^V(g)\chi^V(g^{-1}) \\ &= \frac{1}{|G|}\chi^V(e)\chi^v(e) = |G| \\ &= m_1^2 + m_2^2 + \cdots + m_k^2 \qquad \text{as } X = \bigoplus_i m_i X^{(i)} \end{aligned}$$

■

Proposition: # of irreducible characters = # of conjugacy classes
Proof sketch: Notice that $\chi^V \cong \chi^{\mathbb{C}[G]} \cong \bigoplus_i m_i X^{(i)}$, where $m_i = \chi^{(i)}(e) = deg(X^{(i)}) = dim(V^i)$

$$\begin{aligned} &\implies Com(\mathbb{C}[G]) \cong \mathbb{C}[G] \\ &\implies Z_{\mathbb{C}[G]} \cong Z_{Com(\mathbb{C}[G])} \end{aligned}$$

and so
# of Conjugacy classes = $dim(Z_{\mathbb{C}[G]})$ = # of irreducible representations of G ■
Proposition: For any one dimensional characters χ,

$$\chi(gh) = \chi(g)\chi(h)$$

■

Definition: The *graded trace* of a module $V = \bigoplus_{k\geq 0} V_k$ is the formal power series

$$\chi_q^V = \sum_{k\geq 0} q^k \chi^{V_k}.$$

Example: Let $G = \mathfrak{S}_3$. We look for the graded character of

$$\mathbb{C}[x_1, x_2, x_3] = \mathcal{L}\{1\} \oplus \mathcal{L}\{x_1, x_2, x_3\} \oplus \mathcal{L}\{x_1^2, x_1x_2, x_2^2, \ldots, x_3^2\} \oplus \cdots$$

Solution: We compute the first few terms and find

$$\chi_q^{\mathbb{C}[x_1,x_2,x_3]} = \chi^{(1)} + q(\chi^{(1)} + \chi^{(2)}) + q^2(2\chi^1 + 2\chi^3) + \cdots$$

and then for χ^{V_3}:

$$\begin{aligned} \chi^{V_3}((1)) &= 6 \\ \chi^{V_3}((12)) &= 2 \\ \chi^{V_3}((123)) &= 0 \end{aligned}$$

and hence

$$\begin{aligned} < \chi^{V_3}, \chi^{(1)} > &= 2 \\ < \chi^{V_3}, \chi^{(2)} > &= 0 \\ < \chi^{V_3}, \chi^{(3)} > &= 2 \end{aligned}$$

continuing in this manner, we could conjecture what the graded character is. Instead, we could proceed as follows

$$\begin{aligned} \chi_q^V(e) &= 1 + 3q + 6q^2 + 10q^3 + 15q^4 + \cdots = \frac{1}{(1-q)^3} \\ \chi_q^V((12)) &= 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + \cdots = \frac{1}{(1-q)(1-q^2)} \\ \chi_q^V((123)) &= 1 + q^3 + q^6 + \cdots = \frac{1}{1-q^3} \\ \implies \chi_q^V &= \frac{1}{(1-q)^3}\kappa_1 + \frac{1}{(1-q)(1-q^2)}\kappa_2 + \frac{1}{1-q^3}\kappa_3 \end{aligned}$$

Remark: The multiplicity of $\chi^{(1)}$ in χ_q^V is the Hilbert series $dim_q(\mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3})$ as

$$\begin{aligned} < \chi_q^V, \chi^{(1)} > &= \frac{1}{6}\left(\frac{1}{(1-q)^3} + \frac{1}{(1-q)(1-q^2)} + \frac{1}{(1-q)(1-q^2)}\right) \\ &= \frac{1}{(1-q)(1-q^2)(1-q^3)} \qquad \text{since } \chi^{(1)} = \kappa_1 + \kappa_2 + \kappa_3 \end{aligned}$$

□