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On The Action of the Hall-Littlewood Vertex Operator

A dissertation submitted in partial satisfaction of the
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by

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The dissertation of Michael Zabrocki is approved, and it is acceptable in quality and form for publication on microfilm:

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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ABSTRACT OF THE DISSERTATION

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Professor Adriano Garsia, Chair

The Hall-Littlewood polynomials $H_\mu[X;t] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(t) s_\lambda[X]$ where $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial, are a one parameter family of symmetric functions which form a basis for all symmetric functions. Lascoux and Schützenberger show that there exists a statistic called *charge* on column strict tableaux such that the Hall-Littlewood polynomials are given by the formula

$$H_\mu[X;t] = \sum_{T \in CST^\mu} t^{charge(T)} s_{shape(T)}[X] \qquad (*)$$

There exists a symmetric polynomial operator H_m that has the property that $H_m H_\mu[X;t] = H_{(m,\mu)}[X;t]$ for $m \geq \mu_1$. We give an introduction to Lascoux and Schützenberger's Jeu de Taquin and tableau operators. We describe a combinatorial method for computing the action of the H_m operator on the Schur function basis, $s_\lambda[X]$. This action can then be translated into an operator on column strict tableaux to show how the column strict tableaux of content (m,μ) are created from the column strict tableaux of content μ and to provide an alternate proof of $(*)$.

Chapter 1

Tableaux and Jeu de Taquin

1.1 Standard definitions of partitions and tableau

A partition λ is a weakly decreasing sequence of non-negative integers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$. The length $l(\lambda)$ of the partition is the largest i such that $\lambda_i > 0$. The partition λ is a partition of n if $\lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)} = n$. We associate a partition with its Ferrer's diagram and often use the two interchangeably. We use the French convention and draw the largest part on the bottom of the diagram. One partition is contained in another, $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i (the notation is to suggest that if the diagram for λ were placed over the diagram for μ that one would be contained in the other).

For every partition λ there is a corresponding conjugate partition denoted by λ' where $\lambda'_i =$ the number of cells in the i^{th} column of λ. For example, the conjugate partition of $\lambda = (4, 4, 3, 3, 1)$ is $\lambda' = (5, 4, 4, 2)$.

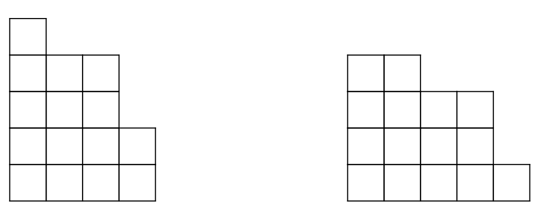
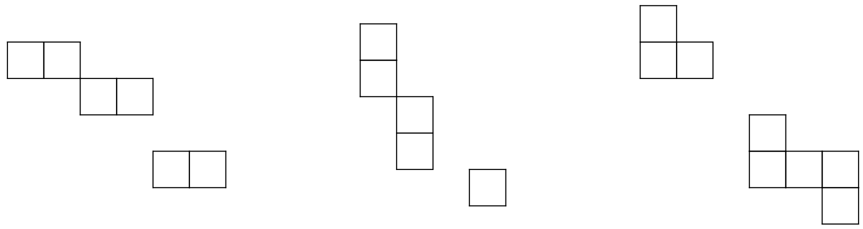


Figure 1.1: The Ferrer's diagrams corresponding to the conjugate partitions $(4, 4, 3, 3, 1)$ and $(5, 4, 4, 2)$.



$(6, 4, 4, 2)/(4, 4, 2)$ $\quad$ $(4, 2, 2, 1, 1)/(3, 1, 1)$ $\quad$ $(6, 6, 4, 3, 2, 1)/(5, 3, 3, 3)$

Figure 1.2: Examples of horizontal, vertical, and border strips

A skew partition is denoted by λ/μ, where it is assumed that $\mu \subseteq \lambda$, and represents the cells that are in λ but are not in μ. A skew partition λ/μ is said to be a horizontal strip if there is at most one cell in each column. Denote the class of horizontal strips of size k by $\mathcal{H}_k$ so that the notation $\lambda/\mu \in \mathcal{H}_k$ means that λ/μ is a horizontal strip with k cells. Similarly, the class of vertical strips (skew partitions with only one cell in each row) will be denoted by $\mathcal{V}_k$ and the class of border strips (skew partitions that contain no 2x2 subpartitions) will be denoted by $\mathcal{B}_k$.

For a fixed partition λ say that a partition ν precedes λ, and use the notation $\nu \to \lambda$, if λ/ν is a single cell. Also say that a partition μ follows λ, and use the notation $\mu \leftarrow \lambda$ if μ/λ is a single cell. The symbol $\nu \to \lambda$ as an index of a summand means that the sum is over all partitions formed by removing a corner cell of λ.

The partitions of n can be given a partial ordering by the definition $\mu \leq \lambda$ if and only if $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all i. This ordering is called the dominance order.

If λ is a partition, then let λ^r denote the partition with the first row removed, that is $\lambda^r = (\lambda_2, \lambda_3, \ldots, \lambda_{l(\lambda)})$. Let λ^c denote the partition with the first column removed, so that $\lambda^c = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{l(\lambda)} - 1)$. This allows us to define the border of a partition μ to be the skew partition μ/μ^{rc}.

Define the k-snake of a partition μ to be the k bottom most right hand cells of the border of μ (the choice of the word "snake" is supposed to suggest the cells that slink with its belly on the ground from the bottom of the partition up along the right hand edge). We use the symbol $ht_k(\mu)$ to denote the height of the k-snake. The symbol $\mu\rfloor_k = (\mu_2 - 1, \mu_3 - 1, \ldots, \mu_h - 1, \mu_1 + h - k - 1, \mu_{h+1}, \ldots \mu_{l(\lambda)})$ will be used to represent a

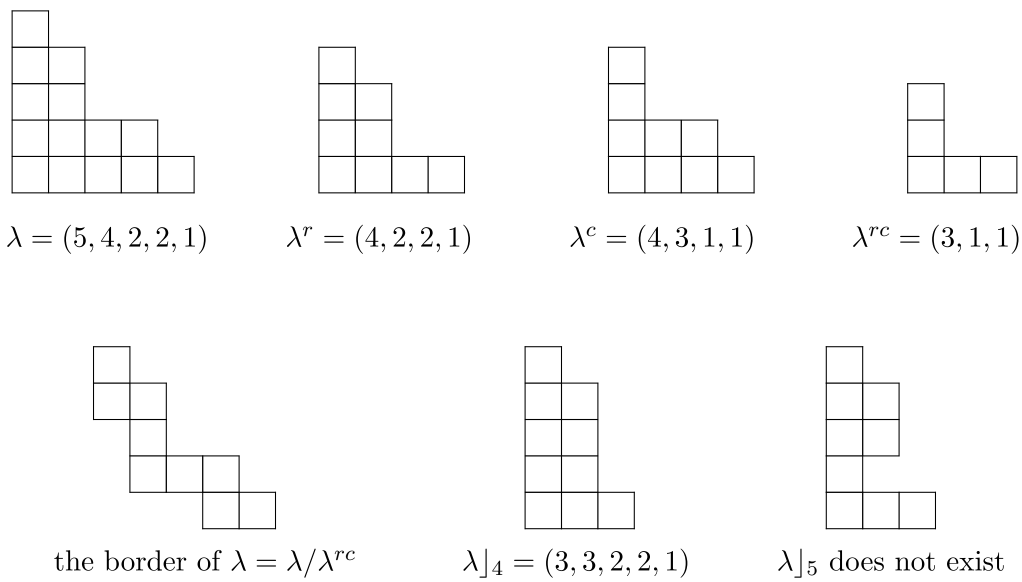


Figure 1.3: Examples of λ, λ^r, λ^c, λ^{rc}, λ/λ^{rc}, and $\lambda\rfloor_n$

partition with the k-snake removed with the understanding that if removing the k-snake does not leave a partition that this symbol is undefined.

If the shape of $\rho = \lambda\rfloor_k$ is given and the height of the k-snake is specified then λ can be recovered (λ is determined from ρ by adding a k-snake of height h). This is because

$$\lambda = (\rho_h + k - h + 1, \rho_1 + 1, \rho_2 + 1, \ldots, \rho_{h-1} + 1, \rho_{h+1}, \rho_{h+2}, \ldots, \rho_{l(\rho)}) \tag{1.1}$$

and so λ will be a partition as long as k is sufficiently large.

A column strict (skew) tableau is a diagram of a partition (or skew partition) with each cell labeled with a positive integer such that the labels increase weakly traveling from left to right in the rows and the labels increase strictly traveling from bottom to top in the columns. All tableaux here are column strict, so a tableau means a column strict tableau.

Let T be a column strict tableau. Denote the shape of the tableau by $\lambda(T)$, the total number of cells in the diagram by $|T|$, and the number of cells labeled with an

i by the symbol T_i. The content of the tableau will be the tuple $\mu(T) = (T_1, T_2, \ldots, T_h)$ (where h is the highest label that appears in the tableau). T is said to be of partition content if the content vector $\mu(T)$ is a partition. The content of a word is defined similarly (the tuple consisting of (the number of 1's in the word, the number of 2's in the word, etc.))

A useful statistic defined on compositions, μ, is $n(\mu) = \sum_i \mu_i(i-1)$.

1.2 Jeu de Taquin

Take a skew tableau of shape λ/μ and let ρ be a partition such that $\rho \leftarrow \lambda$, therefore ρ/λ is a single cell, c. A forward slide into c is an transformation on the tableau. The cell with the largest label that is immediately south or west of c is moved into c creating an empty cell. The newly vacated cell is filled with the contents of the largest label that is south or west of that cell. This continues until the vacated cell $= \nu/\mu$ for some partition $\nu \leftarrow \mu$.

Example 1.2.1. - *perform a forward slide into the cell labeled by c*

$$\begin{array}{ccccccc} 5 & 7 & 8 & c & & & \\ 3 & 3 & 4 & 4 & 5 & 9 & \\ & 1 & 1 & 1 & 2 & 2 & 6 \end{array} \quad \longrightarrow \quad \begin{array}{ccccccc} 3 & 5 & 7 & 8 & & & \\ & 3 & 4 & 4 & 5 & & \\ & 1 & 1 & 1 & 2 & 2 & 6 \end{array}$$

Again consider a skew tableau of shape λ/μ, but now let ν be a partition such that $\nu \to \mu$ and let $c = \mu/\nu$. A reverse slide into c is another transformation on the skew tableau that is just the inverse of the forward slide. The contents of the cell with the smallest label that is immediately to the north or east of c is moved into c leaving an empty cell behind. The newly vacated cell is filled with the contents of the smallest label that is north or east of that cell. This continues until the vacated cell $= \lambda/\rho$ for some partition $\rho \to \lambda$.

Example 1.2.2. - *perform a reverse slide into the cell labeled by c*

$$\begin{array}{lllllllll} 3 & 4 & 5 & 9 & & & & & \\ & & 1 & 1 & 2 & 2 & 3 & 8 & \\ & & & & c & 1 & 2 & 6 & 7 \end{array} \quad \longrightarrow \quad \begin{array}{lllllllll} 3 & 4 & 5 & 9 & & & & & \\ & & 1 & 1 & 2 & 3 & 8 & & \\ & & & & 1 & 2 & 2 & 6 & 7 \end{array}$$

The sequence of cells involved in a forward or reverse slide is called the slide path. An important observation is that slide paths do not cross, that is, if $c1 = \mu/\nu^1$ and $c2 = \nu^1/\nu^2$ are cells such that $\nu^1 \to \mu$ and $\nu^2 \to \nu^1$ then the slide path of $c1$ lies weakly to the left of the slide path of $c2$.

We say that a skew tableau is of straight shape if the shape is a partition. Any skew tableau can be brought to straight shape by applying a sequence of reverse slides until it is of straight shape. There is usually more than one sequence of operations that will bring any one skew tableau to straight shape but is a theorem attributed to both Thomas [T] and Schützenberger [S1] that says the tableau obtained by bringing a skew tableau to straight shape is independent of the sequence of reverse slides.

Example 1.2.3. - *a tableau brought to straight shape under any sequence of reverse slides is the same*

$$\begin{array}{cccccccc} 3&4&5&9&&&& \\ &1&1&2&2&3&8& \\ &&&&1&2&6&7 \end{array} \longrightarrow \begin{array}{cccccccc} 3&4&5&9&&&& \\ &1&1&2&3&8&& \\ &&&1&2&2&6&7 \end{array} \longrightarrow \begin{array}{cccccccc} 3&4&5&9&&&& \\ &1&2&3&8&&& \\ &&1&1&2&2&6&7 \end{array} \longrightarrow$$

$$\begin{array}{cccccccc} 3&4&5&9&&&& \\ &2&3&8&&&& \\ &1&1&1&2&2&6&7 \end{array} \longrightarrow \begin{array}{cccccccc} 3&4&9&&&&& \\ 2&3&5&8&&&& \\ &1&1&1&2&2&6&7 \end{array} \longrightarrow \begin{array}{ccccccc} 3&4&9&&&& \\ 2&3&5&8&&& \\ 1&1&1&2&2&6&7 \end{array}$$

OR

$$\begin{array}{cccccccc} 3&4&5&9&&&& \\ &1&1&2&2&3&8& \\ &&&&1&2&6&7 \end{array} \longrightarrow \begin{array}{cccccccc} 3&4&5&9&&&& \\ 1&1&2&2&3&8&& \\ &&&&1&2&6&7 \end{array} \longrightarrow \begin{array}{ccccccc} 3&4&5&9&&& \\ 1&1&2&2&3&8& \\ &&&1&2&6&7 \end{array} \longrightarrow$$

$$\begin{array}{ccccccc} 3&4&5&9&&& \\ 1&1&2&3&8&& \\ &&1&2&2&6&7 \end{array} \longrightarrow \begin{array}{ccccccc} 3&4&5&9&&& \\ 1&2&3&8&&& \\ &1&1&2&2&6&7 \end{array} \longrightarrow \begin{array}{ccccccc} 3&4&9&&&& \\ 2&3&5&8&&& \\ 1&1&1&2&2&6&7 \end{array}$$

This theorem implies a definition of an equivalence relation. Say that two skew tableau are Jeu de Taquin (JdT) equivalent if they are the same when they are brought to straight shape. This means that applying any number of forward and reverse slides to a tableau leaves it in the same equivalence class.

Concatenation of tableau defines a non-commutative multiplication on tableau under this equivalence relation. If S and T are skew tableau, then define ST to be the skew tableau under the Jeu de Taquin equivalence formed by placing S to the north and west of T. This gives a monoid structure to the set of tableau under JdT equivalence.

1.3 Robinson-Schensted correspondence

Let T be a column strict tableau. There is an operation that inserts a label into the tableau called row insertion. Let x be a non-negative integer label. Replace the entry in the first row that is larger than x and is farthest to the left. Let the y be the displaced label (call y the bumped cell). If no label is in the first row is larger than x then place x at the end of the first row. If x displaces a cell with label y then insert the label y in the second row by either replacing the entry that is larger than y and farthest to the left or by placing it at the end of the second row. Repeat this process until an entry is added to the end of a row.

Example 1.3.1. - *row insert the label* $\boxed{4}$ *into a tableau*

Row insert a $\boxed{4}$ *into*
$$\begin{array}{|c|c|c|c|c|c|c|} \cline{1-3} 3&4&9 \\ \cline{1-4} 2&3&5&8 \\ \hline 1&1&1&2&2&6&7 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|} \cline{1-1} 9 \\ \cline{1-3} 3&4&8 \\ \cline{1-4} 2&3&5&6 \\ \hline 1&1&1&2&2&4&7 \\ \hline \end{array}$$

Knowing the location of the last cell placed in the row insertion algorithm is enough to reverse the process and recover the original tableau and the inserted label. The operation of row deletion is the reverse of row insertion. The operation requires specifying the tableau and a corner cell of the tableau.

If y is the label of a corner cell of T then find the cell on the previous row that is strictly smaller than and farthest to the right. Replace that entry with the label y and let x be the label that was displaced. Find the entry on the previous row that is strictly smaller than x and farthest to the right. Again replace that entry with the label x and continue this way until the first row is reached. The cell from the first row that is displaced is called the ejected cell.

Example 1.3.2. - *row delete the corner cell labeled by* $\boxed{9}$ *in the tableau*

Row delete the topmost corner of the tableau $\begin{ytableau} 3 & 4 & 9 \\ 2 & 3 & 5 & 8 \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{ytableau}$

yields $\begin{ytableau} 3 & 4 \\ 2 & 3 & 5 & 9 \\ 1 & 1 & 1 & 2 & 2 & 6 & 8 \end{ytableau}$ *and ejects a* $\boxed{7}$.

The operation of column insertion and deletion can be similarly defined.

To define column insertion, let T be a column strict tableau and x a non-negative integer label. Replace the entry in the first column that is larger than or equal to x and is farthest to the south. Let the y be the displaced label. If no label is in the first column is larger or equal to x, then place x at the end of the first column. If x displaces a cell with label y, then insert the label y in second column by either replacing the entry that is larger than or equal to y and farthest to the south or by placing it at the end of the first column. Repeat this process until an entry is added to the end of a column.

Example 1.3.3. -*column insert the label* $\boxed{2}$ *into the tableau*

Column insert a $\boxed{2}$ *into* $\begin{ytableau} 3 & 4 & 9 \\ 2 & 3 & 5 & 8 \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{ytableau}$ $\longrightarrow$ $\begin{ytableau} 3 & 4 & 9 \\ 2 & 2 & 3 & 5 & 8 \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{ytableau}$

If the location of the last cell placed in the column insertion algorithm is known then the operation can be inverted. Just as row deletion was defined as the inverse operation to row insertion we can define what it means to column delete a corner cell of a tableau. The operation of column deletion requires selecting a corner cell of the tableau. The result is an ejected label and a tableau with one less cell.

If y is the label of a corner cell of T then find the cell on the previous column that is greater than or equal to and farthest north. Replace that entry with the label y and let x be the label that was displaced. Find the entry on the previous column that is larger than or equal to x and farthest north. Again replace that entry with the label x and continue this way until the first column is reached. The cell from the first column that is displaced is called the ejected cell.

Example 1.3.4. *-column delete the corner cell labeled by* $\boxed{8}$ *in the tableau*

Column deleting the middle corner of the tableau $\begin{array}{ccccccc} 3 & 4 & 9 & & & & \\ 2 & 3 & 5 & 8 & & & \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{array}$

yields $\begin{array}{ccccccc} 4 & 5 & 9 & & & & \\ 2 & 3 & 8 & & & & \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{array}$ *and ejects a* $\boxed{3}$.

The sequence of cells involved in the insertion or deletion process is the bumping path of the insertion or deletion.

1.4 Jeu de Taquin and the Robinson-Schensted correspondence

The operations of row and column insertion can be expressed in terms of JdT operations so that insertion can be recognized as a special case of multiplication in the tableau monoid. The skew tableau with label x placed south and east of a tableau T is Jeu de Taquin equivalent to the tableau produced by row inserting x into T.

This statement is not very obvious until one shows what happens to a single row when a cell with label x is placed south and east. Verify that for a single row, placing a cell south and east is JdT equivalent to the effect of row inserting x into the row (either it is a single row with x at the end or it is a row with x displacing a bumped cell y that is north and west of the row).

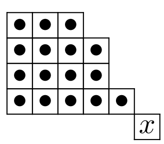
$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & & & & & x \end{array} \longrightarrow \begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & x \end{array} \qquad \text{OR} \qquad \begin{array}{cccccc} y & & & & & \\ \bullet & \bullet & \bullet & x & \bullet & \bullet \end{array}$$

When T is not a single row, it is JdT equivalent to the concatenation of its rows, that is $T = r_k r_{k-1} \ldots r_2 r_1$ where r_i denotes the cells in the i^{th} row of T. Let y_i be the bumped label in the row insertion of the i^{th} row and r'_i be the row after y_{i-1} displaces a cell in r_i, then

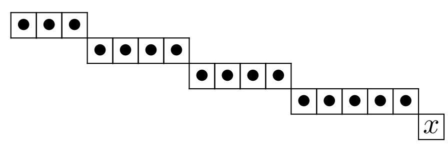
$$
\begin{aligned}
Tx &= r_k r_{k-1} \dots r_2 r_1 x \\
&= r_k r_{k-1} \dots r_2 y_1 r_1' \\
&= r_k r_{k-1} \dots y_2 r_2' r_1' \\
&\vdots \\
&= r_k r_{k-1} \dots r_{j+1} y_j r_i' \dots r_2' r_1'
\end{aligned}
$$

For some j, y_j will be larger than all the cells in r_{j+1} and the resulting rows will form a tableau.

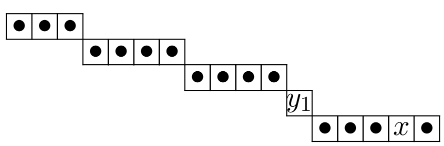
To see this graphically, notice that any tableau is the concatenation of its rows



$\longrightarrow$ is JdT equivalent to



Either there is a label in the first row that is larger than x or x is greater than or equal to the label in every cell in the row so the second of these diagrams is JdT equivalent to either



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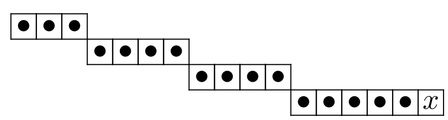
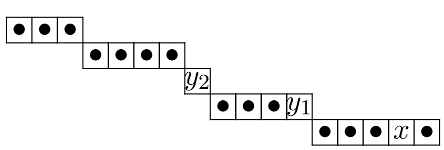


Either there is a label in the second row that is larger than y_1 or y_1 is greater than or equal to the label in every cell in the row so the second of these diagrams is JdT equivalent to either



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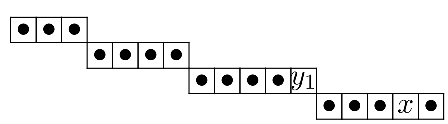


etc.

Breaking down the rows in this way shows that the steps in performing JdT on each of the rows individually is exactly the same as the procedure outlined in the Robinson-Schensted row insertion algorithm.

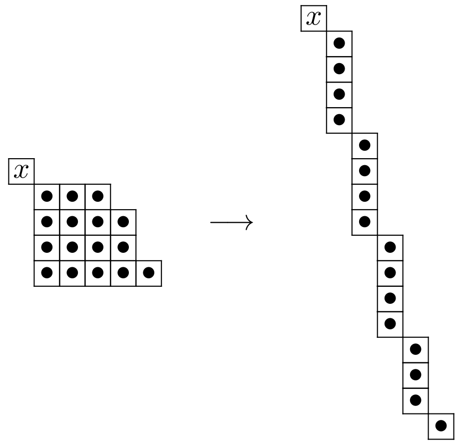


Figure 1.4: A tableau written as the product of its columns.

If the cell x is placed to the north and west of the tableau T, then the resulting straight shape tableau is the same one produced by column inserting the cell x into T, that is xT is JdT equivalent to column inserting x into T.

The justification is similar to the analogous statement for row insertion, first by verifying the statement for single column, and then recognizing that a tableau T can be written as the product of its columns, $T = c_1 c_2 \dots c_k$ where c_i represents the cells in the i^{th} column.

Placing the cell with a label of x to the north and west of the tableau and breaking the tableau into columns makes it easier to see why this operation is equivalent to column insertion. It is not hard to verify they are equivalent when the shape is a single column. For a general tableau, since a tableau can be written as the product of its columns, the figure below shows why a single cell is placed to the north and west has the same effect as the Robinson-Schensted column insertion algorithm.

1.5 Knuth relations

To any skew tableau there is a reading word associated to it formed by reading the entries in the cells in each of the rows from left to right, starting with the top row. The reading word of the tableau T will be denoted by $R(T)$.

There is an equivalence relation on words due to Knuth that is equivalent to the Jeu de Taquin equivalence relation on tableau.

If x, y and z are letters in the words u and v and w and $\tilde{w}$ are subwords then say that two words u and v are elementary Knuth equivalent if either $u = wxzy\tilde{w}$ and $v = wzxy\tilde{w}$ where $x \leq y < z$ or $u = wyzx\tilde{w}$ and $v = wyxz\tilde{w}$ where $x < y \leq z$.

Next say that two words u and v are Knuth equivalent and write $u \sim v$ if they are in the symmetric, transitive, reflexive closure of the elementary Knuth equivalence.

The following theorem shows the relationship between the Knuth equivalence and the JdT equivalence.

Theorem 1.5.1. *(Knuth) Two tableau S and T are Jeu de Taquin equivalent iff $R(T) \sim R(S)$.*

This theorem follows because if $x \leq y < z$ then

$$\begin{array}{|c|cc} \hline x & & \\ \hline & z & \\ & & y \end{array} \longleftrightarrow \begin{array}{|c|c|} \hline z & \\ \hline x & y \\ \hline \end{array} \longleftrightarrow \begin{array}{ccc} z & & \\ & x & \\ & & y \end{array}$$

and if $x < y \leq z$ then

$$\begin{array}{ccc} y & & \\ & z & \\ & & x \end{array} \longleftrightarrow \begin{array}{|c|c|} \hline y & \\ \hline x & z \\ \hline \end{array} \longleftrightarrow \begin{array}{ccc} y & & \\ & x & \\ & & z \end{array}$$

With the operation of concatenation, the words of the positive integer labels forms a monoid. This monoid modulo the Knuth equivalence relation is isomorphic to the monoid of tableau mentioned earlier.

1.6 Charge

There is a statistic defined on words that was introduced by Lascoux and Schützenberger called charge. It is central in the discussion in the remainder of this document.

First, charge is defined for words of content weight $\mu = 1^n$. An index is given to each letter in the word. The index 0 is assigned to 1. If the letter i has index k then the index of the letter $i+1$ is k if $i+1$ lies to the left of i and the index is $k+1$ if $i+1$ lies to the right of i. The charge of the word is defined to be the sum of the indices.

Example 1.6.1. *Let $w = 638152479$ then the index for each letter is given by $\substack{638152479 \\ 213021234}$ and the charge of w is $2+1+3+0+2+1+2+3+4 = 18$.*

If w is a word with content of partition weight then it is first broken up into standard subwords by the following procedure. Place an x under the first 1 in the word traveling from right to left. Next place an x under the first 2 traveling to the left from there. Continue placing an x underneath each of the letters 1 through $l(\mu(w))$ traveling from right to left and beginning again at the right side of the word each time the left hand side is reached. The first standard subword consists of the letters that have xs underneath them read from left to right. Erase these letters to form a new word w' and repeat the procedure forming the next standard subword with the letters 1 through $l(\mu(w'))$. Stop when all letters have been erased. The charge of the word is then defined to be the sum of the charges of the standard subwords.

Example 1.6.2. *Let* $w = 8631412215342795$ *which has content* $\mu(w) = (3, 3, 2, 2, 2, 1, 1, 1, 1)$. *This word has standard subwords and indices given by*

$w^1 = \substack{863215479 \\ 210001123}$ *has charge* 10

$w^2 = \substack{41325 \\ 10112}$ *has charge* 5

$w^3 = \substack{12 \\ 01}$ *has charge* 1

Therefore the charge of w *is* $10 + 5 + 1 = 16$.

Lastly, the charge of a tableau is defined to be the charge of the reading word of the tableau.

Example 1.6.3. *The tableau*

$$T = \begin{array}{|c|c|c|c|c|c|c|}
\cline{1-3} 3 & 4 & 9 \\
\cline{1-4} 2 & 3 & 5 & 8 \\
\hline 1 & 1 & 1 & 2 & 2 & 6 & 7 \\
\hline
\end{array}$$

has a reading word of $R(T) = 34923581112267$ *and the reading word has standard subwords*

$R(T)^1 = \substack{349258167 \\ 015024034}$ *has charge* 19

$R(T)^2 = \substack{312 \\ 101}$ *has charge* 2

$R(T)^3 = \substack{12 \\ 01}$ *has charge* 1

Therefore the charge of the tableau is $19 + 2 + 1 = 22$.

Denote the charge of a tableau, T, by $c(T)$ and of a word, w, by $c(w)$.

It is a theorem that if two words $w \sim w'$ are Knuth equivalent then the charge of the words is equal. This comes first by showing that this is true when w and w' are words with content 1^n and then showing that if $w \sim w'$ then the i^{th} standard subword of w is Knuth equivalent to the i^{th} standard subword of w'. The second of these two statements is not trivial to show and the details may be found in [K] or [Bu].

Considering only tableau of content μ where μ is a partition, the tableau of smallest charge has shape μ with only i's in the i^{th} row. The charge of this tableau will be zero and every other tableau of content μ has shape that is both strictly larger in dominance order and strictly larger in charge than this tableau. A tableau that is only one row high will have maximal charge and the charge will be $n(\mu)$.

Example 1.6.4. *Of the tableaux of content $\mu = (3, 2, 2, 1)$, only the tableau*
$\begin{array}{|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$ *has charge* 0 *and the tableau* $\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ \hline \end{array}$ *has charge* $n(\mu) = 9$.

1.7 Cyclage

There are operations on words and tableau that have the effect of raising and lowering the charge by exactly 1. Let $w = xu$ where x is the first letter of w and u is the remainder of the word, then the cyclage of w is defined to be $w' = ux$. If $w = ux$ where x is the last letter of w and u is the remainder of the word, then the un-cyclage of w is defined to be $w' = xu$.

Let w be a word such that the first letter is not a 1. Let w' be the cyclage of w, the charge of w' is one more than the charge of w (and therefore if a letter that is not 1 is uncyclaged then the charge decreases by 1). This follows directly from the definition of charge since the cyclage operation (as long as the first letter of the word is not 1) only changes one of the standard subwords by moving one letter from the beginning of the word to the end, increasing the index of that letter by 1.

Example 1.7.1. *Consider the word* $w = 8631412215342795$ *has the following standard* *subwords and indices* $\substack{863215479\\210001123}$, $\substack{41325\\10112}$, $\substack{12\\01}$ *and thus the word has charge* $10+5+1=16$.

The cyclage of the word is 6314122153427958 *and this word has the standard subwords and indices* $\substack{632154798\\100011233}$, $\substack{41325\\10112}$, $\substack{12\\01}$ *and thus this word has charge* $11+5+1=17$.

The uncyclage of this word is 5863141221534279 *and this word has the standard subwords and indices* $\substack{863215479\\210001123}$, $\substack{54132\\11011}$, $\substack{12\\01}$ *and thus the word has charge* $10+4+1=15$.

Cyclage on a tableau is defined to be the operation of moving a single cell that is north and west of the rest of the tableau to the south and east of the remainder (move the label from the beginning of the reading word to the end of the reading word).

The operation of cyclage can be defined on a straight shape tableau in terms of column deletion and row insertion.

Since column insertion is equivalent to putting a single cell to the north and west of a tableau and playing Jeu de Taquin, then column deletion is equivalent to playing Jeu de Taquin to move a cell just north and west of the remainder of the tableau.

Placing a cell to the south and east of a tableau and bringing the tableau to straight shape is equivalent to row inserting the cell. Picking a cell up from the north and west of a tableau and placing it to the south and east has the effect of moving the label from the beginning of the reading word to the end.

Example 1.7.2. *Column deleting the corner cell labeled by a* $\boxed{8}$ *in the tableau*

$$\begin{array}{ccccccc} 3 & 4 & 9 & & & & \\ 2 & 3 & 5 & 8 & & & \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{array}$$

produces the tableau

$$\begin{array}{ccccccc} 4 & 5 & 9 & & & & \\ 2 & 3 & 8 & & & & \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{array}$$

and ejects a $\boxed{3}$.

This implies that the tableau

$$\begin{array}{ccccccc} 3 & 4 & 9 & & & & \\ 2 & 3 & 5 & 8 & & & \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{array} \quad \overset{\longrightarrow}{\textit{is JdT equivalent to}} \quad \begin{array}{cccccccc} 3 & & & & & & & \\ & 4 & 5 & 9 & & & & \\ & 2 & 3 & 8 & & & & \\ & 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{array}$$

Example 1.7.3. *Row inserting a the label* $\boxed{3}$ *into the tableau* $\begin{ytableau} 4 & 5 & 9 \\ 2 & 3 & 8 \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{ytableau}$ *yields the tableau*

$$\begin{ytableau} 9 \\ 4 & 5 & 8 \\ 2 & 3 & 6 \\ 1 & 1 & 1 & 2 & 2 & 3 & 7 \end{ytableau}$$

This implies that the skew tableau

$$\begin{ytableau} 4 & 5 & 9 \\ 2 & 3 & 8 \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \\ \none & \none & \none & \none & \none & \none & \none & 3 \end{ytableau} \quad \overrightarrow{\textit{is JdT equivalent to}} \quad \begin{ytableau} 9 \\ 4 & 5 & 8 \\ 2 & 3 & 6 \\ 1 & 1 & 1 & 2 & 2 & 3 & 7 \end{ytableau}$$

Specify a corner cell of a tableau T by picking a shape ν such that ν precedes $\lambda(T)$. Perform a column deletion on the corner cell that corresponds to $\lambda(T)/\nu$ and then row insert the ejected cell. This will be the operation of cyclage (if the corner cell is the highest corner on T then this operation is also called top-cyclage) on a straight shape tableau and it will be denoted by the operator by C_ν that acts on column strict tableaux such that $\lambda(T)$ is a partition and $\lambda(T)/\nu$ is a single cell. The resulting tableau, $C_\nu T$, will also have the property that $\lambda(C_\nu T)/\nu$ is a single cell.

Because this is equivalent to performing cyclage on the reading word of the tableau, this operation has the effect of increasing the charge of the tableau by one as long as the label of the cell that was ejected in the row deletion is not a 1. A shape that is more than one row high will always have at least one corner such that the row deletion will not eject a cell labeled by 1. The top-cyclage on a tableau will always increase the charge by exactly one unless the tableau is one row high.

Example 1.7.4. *Let* $\nu = (7,3,3)$. *To find the cyclage*

$$C_\nu \begin{ytableau} 3 & 4 & 9 \\ 2 & 3 & 5 & 8 \\ 1 & 1 & 1 & 2 & 2 & 6 & 7 \end{ytableau}$$

perform the column deletion on the corner cell labeled by $\boxed{8}$. *Row insert the ejected cell* *label* $\boxed{3}$ *to produce the tableau*

$$\begin{array}{|c|c|c|c|c|c|c|}
\cline{1-1} 9 \\
\cline{1-3} 4 & 5 & 8 \\
\cline{1-3} 2 & 3 & 6 \\
\hline 1 & 1 & 1 & 2 & 2 & 3 & 7 \\
\hline
\end{array}$$

The starting tableau has a reading word 34923581112267 *and charge* 21. *The reading word of the cyclaged tableau is* 94582361112237 *and this has charge* 22.

Uncyclage on a straight shape tableau will be defined to be the reverse of this operation. Pick a corner cell of the tableau and perform row deletion and then column insert the ejected cell. The uncyclage operation will be denoted simply by C_ν^{-1}.

Example 1.7.5. *Again let* $\nu = (7, 3, 3)$. *To find the un-cyclage*

$$C_\nu^{-1} \begin{array}{|c|c|c|c|c|c|c|}
\cline{1-3} 3 & 4 & 9 \\
\cline{1-4} 2 & 3 & 5 & 8 \\
\hline 1 & 1 & 1 & 2 & 2 & 6 & 7 \\
\hline
\end{array}$$

perform the row deletion on the corner cell labeled by $\boxed{8}$. *This ejects a label of* $\boxed{7}$ *which when column inserted into the resulting tableau yields*

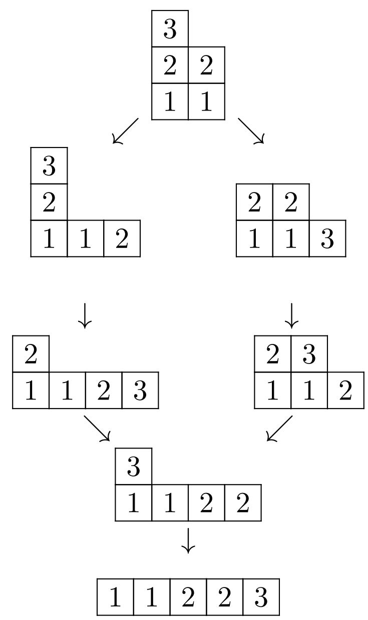
$$\begin{array}{|c|c|c|c|c|c|c|}
\cline{1-1} 7 \\
\cline{1-3} 3 & 4 & 9 \\
\cline{1-3} 2 & 3 & 5 \\
\hline 1 & 1 & 1 & 2 & 2 & 6 & 8 \\
\hline
\end{array}$$

The starting tableau has charge 21 *and the resulting tableau has reading word* 73492351112 268 *and charge* 20.

1.8 The cyclage poset

The set CST^μ forms a ranked poset with the covering relation S covers T if T can be obtained from S by performing one cyclage operation such that the label of the row ejected/column inserted cell is not a 1. The rank of a tableau T in the poset is given by $n(\mu(T)) - c(T)$, this follows because cyclage of a cell that is not a 1 is an invertible operation that increases the charge by 1 and the charge of the bottom element (the tableau of one row) has charge of $n(\mu)$. The rank function on the tableau is called the cocharge of the tableau which will be denoted by $co(T)$. The top element of the poset will be the tableau of shape μ with μ_i i's in the i^{th} row. The minimal element in the poset will be the single row shape of content μ. Notice that any cyclage operation performed on the one row tableau will always cyclage a cell with a minimal label which does not decrease the cocharge.

Example 1.8.1. *The cyclage poset for* $\mu = (2, 2, 1)$

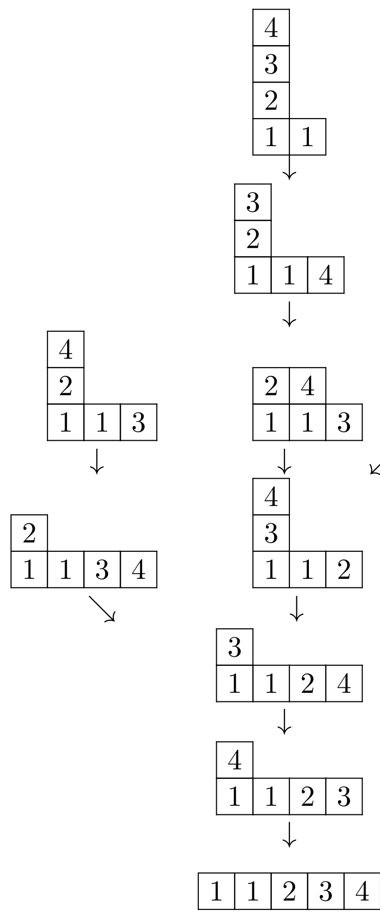


An important property of the cyclage posets is that for $\mu \geq \nu$ in dominance order then there exists an injection $\theta_{\mu\nu}$ from the poset CST^{μ} to CST^{ν} that preserves the cocharge as the rank.

Example 1.8.2. *The cyclage poset for* $\mu = (2, 1, 1, 1)$.

Notice in the poset below that there exists a subposet of that is isomorphic to the cyclage poset for $\mu = (2, 2, 1)$ *such that all the shapes are the same and and the rank agrees, this will be the image of the map* $\theta_{(2,2,1)(2,1,1,1)}$.

In this diagram, an arrow is drawn down for each possible cyclage. In particular, each tableau (except for the one of a single row shape) will have at least one arrow pointing *down for the top cyclage and possibly more arrows if there are other corners such that a column deletion ejects something other than a* 1*.*



The injections $\theta_{\mu\nu}$ will be defined in a later section, but it should be noted now that they do not preserve the charge function, but they do preserve the cocharge function. Charge has the property that $c(T) + co(T) = n(\mu(T))$ and that the $co(T)$ is always the rank in the cyclage poset so that the *cocharge* of a tableau is the number of cyclage operations needed to bring the tableau to a one row shape. This will be the property that will be important when the definition of the charge and cocharge is extended to tableaux that have content weight that is not a partition.

For an arbitrary column strict tableau , T (one that does not necessarily have content that is a partition), define the cocharge to be the number of top-cyclage operations that must be applied to a tableau T so that it is a one row shape. That means that any tableau that is one row high has *cocharge* equal to 0, and the charge of other tableau can be measured by its "distance" from the one row shape. It is a proposition of Lascoux and Schützenberger that the top-cyclage operator is a covering relation for a ranked poset where the bottom element is always the one row shape. A proof of this proposition can be found in [LS1] or in [Br].

We will always define the charge so that $c(T) = n(\mu(T)) - co(T)$, even if $\mu(T)$ is not a partition.

Some operations that are of interest have the property that they preserve the charge or cocharge. The following proposition gives a criterion for preserving the cocharge.

Proposition 1.8.1. *An operator on tableau preserves the cocharge if it commutes with the operation of top-cyclage and the operation on a* 1*-row shape produces a* 1*-row shape. That is, an operator on tableau f preserves cocharge if $fC_\nu T = C_\nu fT$ for all T where ν is the shape that corresponds to top corner of T and f acting on a one row shape is again a one row shape.*

Proof. If the cocharge of T is k then there is a sequence of k top-cyclages that bring it to straight shape. Let $(\nu^1, \nu^2, \ldots, \nu^k)$ be the sequence then
$co(fC_{\nu^k} \cdots C_{\nu^2} C_{\nu^1} T) = 0 = co(C_{\nu^k} \cdots C_{\nu^2} C_{\nu^1} fT) = co(fT) - k$ Thus $co(T) = co(fT) = k$. $\square$

1.9 The action of S_n on the content of tableau and words

There is a definition of a symmetric group action that permutes the content but leaves the shape unchanged, The transposition $(i, i+1)$ will interchange the number of cells labeled with an i and the number of cells that are labeled with an $i+1$ leaving the shape of the tableau unchanged. The action of $(i, i+1)$ is easy to see on the two row shape that has only i's and $i+1$'s by looking at the figure below.

If there are more i's than $i+1$'s ($T_i > T_{i+1}$) in the two row shaped tableau T, then change the rightmost $T_i - T_{i+1}$ i's to $i+1$'s. If there are more $i+1$'s than i's in the two row shaped tableau T, then change $T_{i+1} - T_i$ of the $i+1$'s to i's in the first row.

Example 1.9.1. *Apply the transposition* $(4,5)$ *to a two row shape containing only* 4*'s and* 5*'s*

$$\begin{array}{cccccccccc} 5 & 5 & 5 & 5 & & & & & & \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 \end{array} \xrightarrow{\textit{change 4's to 5's}} \begin{array}{cccccccccc} 5 & 5 & 5 & 5 & & & & & & \\ 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 \end{array}$$

The definition of $(i, i+1)$ on any column strict tableau T is defined by a procedure on the cells in T that are labeled with an i or an $i+1$.

1. Ignore all cells of T that are not labeled with i or $i+1$.
2. Bring the cells labeled with i and $i+1$ to straight shape recording the shape of the skew tableau for each JdT move.
3. Perform the action of $(i, i+1)$ on the tableau that is at most two rows high. This is defined in the procedure above.
4. Perform JdT moves in the reverse order used to bring the cells to straight shape.
5. The tableau with the changes to the cells labeled with i and $i+1$ is the result of $(i, i+1)T$.

Since the transpositions $(i, i+1)$ generate the symmetric group, this defines the action of S_n on the content of the tableau. If $T \in CST^{\mu}$ then $\sigma T \in CST^{\sigma\mu}$ where $\sigma\mu = (\mu_{\sigma(1)}, \mu_{\sigma(2)}, \ldots, \mu_{\sigma(l(\mu))})$.

Example 1.9.2. *The tableau* $T = \begin{array}{cccc} 6 & & & \\ 4 & 4 & & \\ 2 & 3 & 7 & \\ 1 & 2 & 5 & 8 \end{array}$ *has content* $\mu(T) = (1,2,1,2,1,1,1,1)$. *To act on by the transposition* $(4,5)$ *perform the following steps.*

$$(4,5)\begin{array}{cccc} 6 & & & \\ 4 & 4 & & \\ 2 & 3 & 7 & \\ 1 & 2 & 5 & 8 \end{array} \xrightarrow{\textit{ignore all but 4's and 5's}} \begin{array}{ccc} 4 & 4 & \\ & & \\ & & 5 \end{array}$$

$$\xrightarrow{\textit{bring to straight shape}} \begin{array}{ccc} 4 & 4 & 5 \end{array}$$

$$\xrightarrow{\textit{exchange the number of 4's and 5's}} \begin{array}{ccc} 4 & 5 & 5 \end{array}$$

$$\xrightarrow[\textit{return cells to original positions}]{} \quad \begin{array}{cccc} & \boxed{4} & \boxed{5} & \\ & & & \\ & & & \boxed{5} \end{array}$$

$$\xrightarrow[\textit{restore remainder of tableau}]{} \quad \begin{array}{|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-3} 2 & 3 & 7 \\ \hline 1 & 2 & 5 & 8 \\ \hline \end{array}$$

This definition does require some justification because it is not clear that $(i, i+1)$ is well defined (there are many choices that are made when bringing the cells labeled with an i or $i+1$ to straight shape or that the reverse JdT slides will produce the same shapes), for this we refer to [Fu] appendix A.3.

It also must be checked that the transpositions satisfy the coxeter relations to verify that they generate the symmetric group. The fact that $(i, i+1)^2 = 1$ and $(i, i+1)(j, j+1) = (j, j+1)(i, i+1)$ if $|i - j| \geq 2$ are clear from the definition. The relation $(i, i+1)(i+1, i+2)(i, i+1) = (i+1, i+2)(i, i+1)(i+1, i+2)$ is not as clear and there is something to check.

The S_n action on tableaux preserves cocharge. To verify this it must be checked that $(i, i+1)$ commutes with cyclage. Since the effect of $(i, i+1)$ leaves the cocharge fixed then it must be that the charge has the property that $c((i, i+1)T) = n(\mu((i, i+1)T)) - co((i, i+1)T) = T_i - T_{i+1} + n(\mu(T)) - co(T) = T_i - T_{i+1} + c(T)$.

1.10 The injection $\theta_{\mu\nu} : CST^{\mu} \to CST^{\nu}$

As was mentioned in previous sections, if $\mu \geq \nu$ in dominance order then there exists a cocharge preserving injection, $\theta_{\mu\nu}$ that maps the column strict tableau of content μ to the column strict tableau of content ν. This result is a theorem of Lascoux and Schützenberger. In addition, they show that if $\mu \geq \nu \geq \omega$ then $\theta_{\mu\omega} = \theta_{\mu\nu} \circ \theta_{\nu\omega}$.

By design this map will preserve cocharge and the effect on the charge will depend on μ and ν. The relation is given by the dependence

$$c(\theta_{\mu\nu}T) = n(\nu) - co(\theta_{\mu\nu}T) = n(\nu) - co(T) = n(\nu) - n(\mu) + c(T)$$

To compute the map $\theta_{\mu\nu}$ on a column strict tableau T, find the sequence of shapes of top-cyclages that bring T to a one row shape $R_\mu = C_{\nu_k} \cdots C_{\nu_2} C_{\nu_1} T$. The image of the one row shape of content μ is the one row shape of content ν, R_ν. Perform the un-cyclages corresponding to the cyclages used to bring T to a one row shape and let $\theta_{\mu\nu} T = C_{\nu_1}^{-1} C_{\nu_2}^{-1} \cdots C_{\nu_k}^{-1} R_\nu$.

Example 1.10.1. *The image of* $\begin{array}{|c|c|c|}\cline{1-2} 2&2&\multicolumn{1}{c}{}\\\hline 1&1&1\\\hline\end{array}$ *under the maps* $\theta_{(3,2)(2,2,1)}$*,* $\theta_{(3,2)(2,1,1,1)}$*, and* $\theta_{(3,2)(1,1,1,1,1)}$ *can be computed by the following diagram.*

$$
\begin{array}{ccccccc}
\begin{array}{|c|c|c|}\cline{1-2} 2&2&\multicolumn{1}{c}{}\\\hline 1&1&1\\\hline\end{array} & & \begin{array}{|c|c|c|}\cline{1-2} 2&3&\multicolumn{1}{c}{}\\\hline 1&1&1\\\hline\end{array} & & \begin{array}{|c|c|c|}\cline{1-2} 3&4&\multicolumn{1}{c}{}\\\hline 1&1&2\\\hline\end{array} & & \begin{array}{|c|c|c|}\cline{1-2} 4&5&\multicolumn{1}{c}{}\\\hline 1&2&3\\\hline\end{array} \\
C_{(3,1)} \downarrow & & C_{(3,1)}^{-1} \uparrow & & C_{(3,1)}^{-1} \uparrow & & C_{(3,1)}^{-1} \uparrow \\
\begin{array}{|c|c|c|c|}\cline{1-1} 2&\multicolumn{3}{c}{}\\\hline 1&1&1&2\\\hline\end{array} & & \begin{array}{|c|c|c|c|}\cline{1-1} 3&\multicolumn{3}{c}{}\\\hline 1&1&2&2\\\hline\end{array} & & \begin{array}{|c|c|c|c|}\cline{1-1} 4&\multicolumn{3}{c}{}\\\hline 1&1&2&3\\\hline\end{array} & & \begin{array}{|c|c|c|c|}\cline{1-1} 5&\multicolumn{3}{c}{}\\\hline 1&2&3&4\\\hline\end{array} \\
C_{(4)} \downarrow & & C_{(4)}^{-1} \uparrow & & C_{(4)}^{-1} \uparrow & & C_{(4)}^{-1} \uparrow \\
\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline\end{array} & \to & \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline\end{array} & \to & \begin{array}{|c|c|c|c|c|}\hline 1&1&2&3&4\\\hline\end{array} & \to & \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline\end{array}
\end{array}
$$

Therefore we have that

$$\theta_{(3,2)(2,2,1)} \begin{array}{|c|c|c|}\cline{1-2} 2&2&\multicolumn{1}{c}{}\\\hline 1&1&1\\\hline\end{array} = \begin{array}{|c|c|c|}\cline{1-2} 2&3&\multicolumn{1}{c}{}\\\hline 1&1&1\\\hline\end{array}$$

$$\theta_{(3,2)(2,1^3)} \begin{array}{|c|c|c|}\cline{1-2} 2&2&\multicolumn{1}{c}{}\\\hline 1&1&1\\\hline\end{array} = \begin{array}{|c|c|c|}\cline{1-2} 3&4&\multicolumn{1}{c}{}\\\hline 1&1&2\\\hline\end{array}$$

$$\theta_{(3,2)(1^5)} \begin{array}{|c|c|c|}\cline{1-2} 2&2&\multicolumn{1}{c}{}\\\hline 1&1&1\\\hline\end{array} = \begin{array}{|c|c|c|}\cline{1-2} 4&5&\multicolumn{1}{c}{}\\\hline 1&2&3\\\hline\end{array}$$

1.11 Insertion and deletion of 1s in a tableau

The operation of deleting the first n cells labeled with 1s on a tableau T of content $(m+n, \mu)$ and then playing Jeu de Taquin to bring the tableau to straight shape does not change the charge or cocharge if $m \geq \mu_1$. The charge remains the same because in the word definition of charge killing the n 1s only deletes standard subwords of length one and charge 0. The cocharge remains the same because the charge is the same and $n(m, \mu) = n(m+n, \mu)$. Let $K_n(T)$ denote the tableau produced by this operation (K_n

stands for kill n cells labeled by 1). K_n is a surjection from the tableau of content $(\mu_1+n, \mu_2, \ldots, \mu_k)$ to the tableau of content μ.

$K_n(T)$ has the property that $\lambda(T)/\lambda(K_n(T))$ is a horizontal strip and $c(T) = c(K_n(T))$. $\lambda(T)/\lambda(K_n(T))$ will be a horizontal strip because the n 1s that are deleted are a horizontal strip and their paths under the Jeu de Taquin operation will keep them as a horizontal strip as cells travel to the outside of the shape. The reason the charge does not change is that the skew tableau of T with the extra 1s removed has the same charge as T and Jeu de Taquin operations on that skew tableau does not change the charge.

Example 1.11.1. *Apply K_6 to the tableau below of content* $(9,3,2,1,1,1,1,1)$

$$K_6\begin{array}{cccccccccc} 6 \\ 3 & 5 & 7 \\ 2 & 2 & 2 & 3 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{array} = \begin{array}{ccccc} 6 \\ 3 & 5 & 7 \\ 2 & 2 & 2 & 8 \\ 1 & 1 & 1 & 3 & 4 \end{array}$$

both of these tableau have charge 9. *The resulting tableau has content* $(3,3,2,1,1,1,1,1)$.

Starting with a tableau T of content (m,μ) (with $m \geq \mu_1$) and a partition λ of $m+n+|\mu|$ such that $\lambda/\lambda(T)$ is a horizontal strip, it is possible to add n 1s to T in an inverse operation to K_n under certain conditions. Denote the operator that adds the n 1s to the tableau in this way by A_n^{λ}. $A_n^{\lambda}T$ will exist if sliding the cells of T into a skew tableau of shape $\lambda/(n)$ (the cells in $\lambda/\lambda(T)$ must be filled from left to right) moves the m 1s in the skew tableau only to the right. Because JdT sliding paths do not cross we note that we need only check that filling the leftmost cell of $\lambda/\lambda(T)$ by playing Jeu de Taquin on T moves the 1s in T to the right.

Example 1.11.2. $A_2^{(5,4,2)}\begin{array}{cccc} 5 & 6 \\ 2 & 2 & 4 \\ 1 & 1 & 1 & 3 \end{array}$ *does not exist because playing JdT on the tableau to create a skew tableau of shape* $(5,4,2)/(2)$ *yields*

$$\begin{array}{ccccc} 5 & 6 \\ 2 & 2 & 4 & \bullet \\ 1 & 1 & 1 & 3 & \bullet \end{array} \longrightarrow \begin{array}{ccccc} 5 & 6 \\ 1 & 2 & 2 & 4 \\ & & 1 & 1 & 3 \end{array}$$

and the 1s in the tableau did not move only to the right in this operation.

Example 1.11.3. *To find the action of* $A_4^{(9,7,4,2)}$ *on the tableau*

$$\begin{array}{ccccccc} 4 & 5 & & & & & \\ 3 & 4 & 6 & 7 & & & \\ 2 & 2 & 2 & 5 & 8 & & \\ 1 & 1 & 1 & 3 & 6 & 7 & 8 \end{array}$$

first play JdT on the tableau to change it to skew shape $(9,7,4,2)/(4)$*.*

$$\begin{array}{ccccccccc} 4 & 5 & & & & & & & \\ 3 & 4 & 6 & 7 & & & & & \\ 2 & 2 & 2 & 5 & 8 & \bullet & \bullet & & \\ 1 & 1 & 1 & 3 & 6 & 7 & 8 & \bullet & \bullet \end{array} \longrightarrow \begin{array}{ccccccccc} 4 & 5 & & & & & & & \\ 3 & 4 & 6 & 7 & & & & & \\ 2 & 2 & 2 & 5 & 6 & 8 & 8 & & \\ & & & & 1 & 1 & 1 & 3 & 7 \end{array}$$

so then

$$A_4^{(9,7,4,2)} \begin{array}{ccccccc} 4 & 5 & & & & & \\ 3 & 4 & 6 & 7 & & & \\ 2 & 2 & 2 & 5 & 8 & & \\ 1 & 1 & 1 & 3 & 6 & 7 & 8 \end{array} = \begin{array}{ccccccccc} 4 & 5 & & & & & & & \\ 3 & 4 & 6 & 7 & & & & & \\ 2 & 2 & 2 & 5 & 6 & 8 & 8 & & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 7 \end{array}$$

Notice that reversing these operations is the same as applying K_4 *to the last tableau.*

1.12 Raising the content of a tableau

Form a tableau by changing all the 1's in T to 2's, all the 2's in T to 3's, etc. Denote this type of change to the content by $(T\uparrow)$. Place the resulting tableau on a row of m 1's (where $m \geq \lambda(T)_1$) to make a tableau, $\mathbf{S}_m T$, of shape $\lambda(\mathbf{S}_m T) = (m, \lambda(T))$ and content $\mu(\mathbf{S}_m T) = (m, \mu(T))$.

$\mathbf{S}_m T$ will have charge equal to the charge of T. This follows simply because each standard subword w_i of $R(T)$ corresponds to a standard subword of $R(\mathbf{S}_m T)$, $(w_i\uparrow)1$. The definition of charge gives that $c(w_i) = c((w_i\uparrow)1)$. The other standard subwords of $R(\mathbf{S}_m T)$ will be just 1 and have charge 0. The cocharge of $\mathbf{S}_m T$ will be $|T|$ higher than the cocharge of T because $n(\mu(\mathbf{S}_m T)) = n(m, \mu) = |T| + n(\mu(T))$.

This procedure will be one of the steps for creating tableaux of content (m, μ) from the tableaux of content μ.

Example 1.12.1. *Start with the tableau* $T = \begin{array}{ccc} 2 & 3 & \\ 1 & 1 & 2 \end{array}$ *with charge* 2 *and cocharge* 2*. If* *we let $m = 3$ then the corresponding tableau $\mathbf{S}_m T$ is*

$$\begin{array}{|c|c|c}\cline{1-2} 3 & 4 & \\ \hline 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

This tableau has charge 2 and cocharge 7.

There is also a way of raising the content by adding 1's without changing the cocharge and increasing the charge only. Again create a tableau $(T \uparrow)$ by changing the 1s in T to 2s, 2s to 3s, etc. Next, slide the cells in the first row to the right by m spaces and fill the space with 1s. This produces a new tableau $\mathbf{R}_m T$ that is of shape $\lambda(\mathbf{R}_m T) = (m + \lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$ and has content $\mu(\mathbf{R}_m T) = (m, \mu(T))$.

$\mathbf{R}_m T$ will have cocharge equal to the cocharge of T because the charge will increase by $|T|$. Tracing the algorithm of charge it is easy to see that $c(\mathbf{R}_m T) = c(T) + |T|$. Again, because of the relationship that $c(\mathbf{R}_m T) + co(\mathbf{R}_m T) = n(\mu(\mathbf{R}_m T)) = n(\mu) + |T| = c(T) + co(T) + |T|$, then it must be that $co(\mathbf{R}_m T) = co(T)$

Example 1.12.2. *Start with the tableau $T = \begin{array}{|c|c|c}\cline{1-2} 2 & 3 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$ with charge 2 and cocharge 2. If we let $m = 3$ then the corresponding tableau $\mathbf{R}_m T$ is*

$$\begin{array}{|c|c|cccc}\cline{1-2} 3 & 4 & & & & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

This tableau has charge 7 and cocharge 2.

Chapter 2

Symmetric Functions

2.1 Symmetric functions

The symmetric polynomials of S_n, denoted by Γ_n is a subring of $\mathbb{Q}[x_1, x_2, \ldots x_n]$ consisting of those polynomials that are invariant when the elements of the symmetric group S_n act on the polynomial by permuting the variables, that is $P(x_1, x_2, \ldots, x_n)$ is a symmetric polynomial iff $P(x_1, x_2, \ldots, x_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for all $\sigma \in S_n$.

For $m \geq n$ there is a natural embedding of Γ_m into Γ_n by setting the variables $x_{n+1}, \ldots, x_m$ to 0. The number of variables in a symmetric function is usually irrelevant, just that it is large enough for the formulas that are used. It is convenient to work with symmetric polynomials in an infinite number of variables, but these objects are no longer polynomials, but formal infinite sums of monomials.

Here we will consider the ring of symmetric functions in an infinite number of variables as a subring of $\mathbb{Q}[x_1, x_2, \ldots]$. A more precise construction of this ring can be found in [M] section I.2.

2.2 Plethystic notation

We make use of plethystic notation for symmetric functions here. This is a notational device for expressing the substitution of the monomials of one expression, $E = E(t_1, t_2, t_3, \ldots)$ for the variables of a symmetric function, P. The result will be denoted by $P[E]$ and represents the expression found by expanding P in terms of the power symmetric functions and then substituting for p_k the expression $E(t_1^k, t_2^k, t_3^k, \ldots)$.

More precisely, if the power sum expansion of the symmetric function P is given by

$$P = \sum_\lambda c_\lambda p_\lambda$$

then the $P[E]$ is given by the formula

$$P[E] = \sum_\lambda c_\lambda p_\lambda \Big|_{p_k \to E(t_1^k, t_2^k, t_3^k, \ldots)}$$

.

To express a symmetric function in a single set of variables $x_1, x_2, \ldots, x_n$, let $X_n = x_1 + x_2 + \cdots + x_n$. The expression $P[X_n]$ represents the symmetric function P evaluated at the variables $x_1, x_2, \ldots, x_n$ since

$$P(x_1, x_2, \ldots, x_n) = \sum_\lambda c_\lambda p_\lambda \Big|_{p_k \to x_1^k + x_2^k + \cdots + x_n^k} = P[X_n]$$

The Cauchy kernel is a ubiquitous formula in the theory of symmetric functions (especially when working with plethystic notation). We introduce it here and state a few of its properties since they will be used in most of the symmetric function identities further on.

Definition 2.2.1. *The Cauchy kernel*

$$\Omega[X] = \prod_i \frac{1}{1 - x_i}$$

It follows using plethystic notation that $\Omega[X]\Omega[Y] = \Omega[X + Y]$ and $\Omega[-X] = \prod_i (1 - x_i)$.

2.3 The classical bases- p_λ, s_λ, e_λ, h_λ, m_λ and the Pieri rule

The following definitions of symmetric functions are standard and may be found in any text on the subject of symmetric functions.

The monomial symmetric function indexed by a partition λ will be denoted by m_λ is given by the formula

$$m_\lambda[X] = \sum_\alpha x^\alpha$$

summed over α that are distinct rearrangements of the partition λ.

For each k there is the k^{th} power symmetric function given by the formula

$$p_k[X] = \sum_{i\geq 0} x_i^k$$

For each partition λ we set the power symmetric function indexed by λ to be $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_{l(\lambda)}}$.

For each $k \geq 0$ there exists the k^{th} homogeneous symmetric function is given by the formula

$$h_k[X] = \sum_{i_1\leq i_2\leq\cdots\leq i_k} x_{i_1}x_{i_2}\cdots x_{i_k} = \prod_{i\geq 0}\frac{1}{(1-x_it)}\Big|_{t^k}$$

Again for each partition we set the homogeneous symmetric function indexed by λ to be the product $h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_{l(\lambda)}}$ and hence h_λ is given by the formula

$$h_\lambda[X] = \prod_{k=1}^{l(\lambda)}\prod_{i\geq 0}\frac{1}{1-x_iz_k}\Big|_{z_1^{\lambda_1}z_2^{\lambda_2}\ldots z_{l(\lambda)}^{\lambda_{l(\lambda)}}} = \Omega[XZ]\Big|_{z_1^{\lambda_1}z_2^{\lambda_2}\ldots z_{l(\lambda)}^{\lambda_{l(\lambda)}}}$$

There also exists the k^{th} elementary symmetric function given by the formula

$$e_k[X] = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k} = \prod_{i\geq 0}(1+x_it)\Big|_{t^k}$$

Again we set for each partition λ, the elementary symmetric function indexed by λ is the product $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_{l(\lambda)}}$

The Schur function indexed by a partition λ in a finite number of variables is usually given by the ratio of two alternating polynomials. Let

$$a_p(x_1,\ldots,x_n) = det(x_j^{p_i}) = \sum_{\sigma\in S_n}\epsilon(\sigma)x_1^{p_{\sigma(1)}}x_2^{p_{\sigma(2)}}\cdots x_n^{p_{\sigma(n)}}$$

This is an alternating polynomial that $= 0$ if $p_i = p_j$ for some $i \neq j$. When $p = \delta = (n-1, n-2, \ldots, 1, 0)$ then $a_\delta(x_1,\ldots,x_n) = det(x_j^{n-i}) = \prod_{i<j}(x_i - x_j)$ is called the Vandermonde determinant. For any partition λ, define the Schur polynomial to be the ratio

$$a_{\lambda+\delta}(x_1,\ldots,x_n)/a_\delta(x_1,\ldots,x_n)$$

Here we wish to define the Schur symmetric function in a countable set of variables so that the similarities to the homogeneous and Hall-Littlwood symmetric functions are clearer. We will use the following definition

Definition 2.3.1. *The Schur symmetric functions*

$$s_\lambda[X] = \Omega[XZ] \prod_{1 \le i < j \le k} 1 - z_j/z_i \Big|_{Z^\lambda}$$

where $Z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_k^{\lambda_k}$.

The Cauchy kernel evaluated at the product of two sets of variables has the formula ([M] p 63)

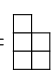
$$\Omega[XY] = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda[X] s_\lambda[Y] = \sum_\lambda h_\lambda[X] m_\lambda[Y]$$

There exists combinatorial rules for computing the products of Schur functions with elementary and homogeneous symmetric functions with only one part. The Pieri formula ([M] p 73) for the product of $h_m[X]$ and $s_\lambda[X]$ is given by

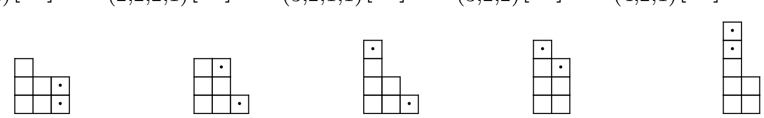
$$h_m[X] s_\lambda[X] = \sum_{\mu/\lambda \in \mathcal{H}_m} s_\mu[X] \tag{2.1}$$

and the rule for the product of $e_m[X]$ and $s_\lambda[X]$ is given by

$$e_m[X] s_\lambda[X] = \sum_{\mu/\lambda \in \mathcal{V}_m} s_\mu[X] \tag{2.2}$$

Example 2.3.1. $\lambda =$  $= (2, 2, 1)$

$$h_2[X] s_{(2,2,1)}[X] = s_{(2,2,2,1)}[X] + s_{(3,2,1,1)}[X] + s_{(3,2,2)}[X] + s_{(4,2,1)}[X]$$



$$e_2[X] s_{(2,2,1)}[X] = s_{(3,3,1)}[X] + s_{(3,2,2)}[X] + s_{(3,2,1,1)}[X] + s_{(2,2,2,1)}[X] + s_{(2,2,1,1,1)}[X]$$

There exists operators $h_m^\perp$ and $e_m^\perp$ which are 'dual' to multiplication by the corresponding symmetric function. Instead of introducing the inner product space structure on the symmetric functions, here we define these two operators by their action on the Schur function basis and give their combinatorial definition.

Definition 2.3.2. *The dual homogeneous operator*

$$h_m^{\perp} s_\lambda[X] = \sum_{\mu:\lambda/\mu \in \mathcal{H}_m} s_\mu[X]$$

Definition 2.3.3. *The dual elementary operator*

$$e_m^{\perp} s_\lambda[X] = \sum_{\mu:\lambda/\mu \in \mathcal{V}_m} s_\mu[X]$$

Sometimes these operators will be expressed with an X (for example, $e_j^{X\perp}$) to make clear that they are acting on the symmetric functions in the X set of variables only, but when it is clear which symmetric functions are relevant the X will be left off.

The expression $\Omega[XY]$ is sometimes called a reproducing kernel because of the following property

Lemma 2.3.1.

$$e_j^{X\perp}\Omega[XZ] = e_j[Z]\Omega[XZ]$$

$$h_j^{X\perp}\Omega[XZ] = h_j[Z]\Omega[XZ]$$

Proof. (Outline) These identities follow from the identity that $\Omega[XZ] = \sum_\lambda s_\lambda[X]s_\lambda[Z]$ then use the Pieri formulas (2.1) (2.2) and the combinatorial definition of $e_j^{X\perp}$ and $h_j^{X\perp}$ (def 2.3.2) (def 2.3.3) and equate coefficients of $s_\lambda[X]$ on both sides of the equation. □

2.4 The Schur function vertex operator and combinatorial action

We are interested in vertex operators for symmetric functions, that is, operators which add a row (or a column) to a partition which indexes the symmetric function. The vertex operator that adds a row of size m to the homogeneous and elementary symmetric functions is just multiplication by the m^{th} homogeneous or elementary symmetric function. Multiplication by h_m can be written to give a formula of a vertex operator flavor:

$$h_m h_\lambda[X] = h_{(m,\lambda)}[X]$$

We wish to understand vertex operators by looking at their action on the Schur function basis. The action of the "homogeneous" (and "elementary") vertex operator on the Schur function basis is the well known Pieri formula. There exists other vertex operators, and in this section we would like to examine the vertex operator for the Schur function basis. This operator was introduced by J. N. Bernstein (see [Z] p. 69 or [M] p. 96) and for $m \geq 0$ is given by the following formula

$$S_m = \sum_{i \geq 0} h_{m+i}[X] e_i^{\perp} (-1)^i$$

This operator may also be expressed plethystically by the formula

$$S_m P[X] = P\left[X - \frac{1}{z}\right] \Omega[zX]\Big|_{z^m}$$

and it has the property that if $m \geq \lambda_1$ then

$$S_m s_\lambda[X] = s_{(m,\lambda)}[X] \tag{2.3}$$

This property follows easily from the definition of the Schur symmetric function and the plethystic notation of the operator S_m, since

$$S_m s_\lambda[X] = S_m \Omega[XZ] \prod_{1 \leq i < j \leq k} 1 - z_j/z_i \Big|_{Z^\lambda} \tag{2.4}$$

$$= \Omega\left[\left(X - \frac{1}{z_o}\right) Z\right] \Omega[z_o X]\Big|_{z_o^m} \prod_{1 \leq i < j \leq k} 1 - z_j/z_i \Big|_{Z^\lambda} \tag{2.5}$$

$$= \Omega[X(z_o + Z)] \prod_{j=1}^{k} (1 - z_j/z_o) \prod_{1 \leq i < j \leq k} 1 - z_j/z_i \Big|_{z_o^m Z^\lambda} \tag{2.6}$$

$$= \Omega[X(z_o + Z)] \prod_{0 \leq i < j \leq k} 1 - z_j/z_i \Big|_{z_o^m Z^\lambda} = s_{(m,\lambda)}[X] \tag{2.7}$$

The generating function for the Schur function vertex operator will be denoted by $\mathbf{S}(z)$ and it may be expressed by

$$\mathbf{S}(z) P[X] = \sum_m z^m S_m P[X] = P\left[X - \frac{1}{z}\right] \Omega[zX]$$

Notice that on a symmetric function $P[X]$ we have that

$$\begin{aligned}
\mathbf{S}(z_1)\mathbf{S}(z_2)P[X] &= \mathbf{S}(z_1)P\left[X - \frac{1}{z_2}\right]\Omega[z_2X] \\
&= P\left[X - \frac{1}{z_1} - \frac{1}{z_2}\right]\Omega\left[z_2\left(X - \frac{1}{z_1}\right)\right]\Omega[z_1X] \\
&= P\left[X - \frac{1}{z_1} - \frac{1}{z_2}\right]\Omega[z_2X]\Omega[z_1X](1 - z_2/z_1) \\
&= \frac{1 - z_2/z_1}{1 - z_1/z_2}\mathbf{S}(z_2)\mathbf{S}(z_1)P[X] \\
&= -\frac{z_2}{z_1}\mathbf{S}(z_2)\mathbf{S}(z_1)P[X]
\end{aligned}$$

By equating coefficients of $z_1^m z_2^n$ in this formula we have that

$$S_mS_n = -S_{n-1}S_{m+1} \tag{2.8}$$

and by setting $n = m+1$ we see also that $S_mS_{m+1} = 0$.

This commutation relation gives us the following proposition using the notation introduced in chapter 1:

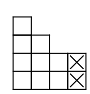
Proposition 2.4.1. *Let m be a non-negative integer and λ a partition of n. Choose k such that $m + k \geq \lambda_1$, then*

$$S_m s_\lambda[X] = (-1)^{ht_k((m+k,\lambda))-1} s_{(m+k,\lambda)\rfloor_k}[X]$$

and $S_m s_\lambda[X] = 0$ if removing the k-snake from $(m+k,\lambda)$ does not leave a partition.

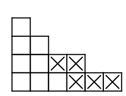
Example 2.4.1.

$$S_2 s_{(4,2,1)}[X] = -s_{(3,3,2,1)}[X]$$



using $k = 2$

$$S_1 s_{(4,2,1)}[X] = -s_{(3,2,2,1)}[X]$$



using $k = 5$

Proof. By the definition, $(m+k,\lambda)\rfloor_k = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{h-1} - 1, (m+k) - k + h - 1, \lambda_h, \ldots \lambda_{l(\lambda)})$ where h is the height of the k-snake in $(m+k,\lambda)$. By the property 2.3 we have that $S_m s_\lambda[X] = S_m S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_{l(\lambda)}} 1$. Apply the relation 2.8 then there will exist an h' such that either $m + h' = \lambda_{h'}$ or $m + h' - 1 \geq \lambda_{h'}$ and

$$S_m s_\lambda[X] = (-1)^{h'-1} S_{\lambda_1 - 1} S_{\lambda_2 - 1} \cdots S_{\lambda_{h'-1} - 1} S_{m+h'-1} S_{\lambda_{h'}} \cdots S_{\lambda_{l(\lambda)}} 1 \qquad (2.9)$$

Now if $m + h' = \lambda_{h'}$ then $S_m s_\lambda[X] = 0$ since $S_{m+h'-1} S_{\lambda_{h'}} = 0$. Adding a k-snake of height h' to $(\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_{h'-1} - 1, m + h' - 1, \lambda_{h'}, \cdots, \lambda_{l(\lambda)})$ yields $(m+k, \lambda)$ and so it must be that $h = h'$ and $S_m s_\lambda[X] = (-1)^{ht_k(\lambda)-1} s_{(m+k,\lambda)\rfloor_k}[X]$. $\square$

This proposition gives a combinatorial method for finding the action of the Schur function vertex operator when m is less than λ_1. When $m \geq \lambda_1$, S_m simply adds a row of length m to the Schur function $s_\lambda[X]$.

There is also an operator $\tilde{S}_m$ which adds a column of length m to a Schur function. $\tilde{S}_m = \omega S_m \omega$ where the involution ω sends $s_\lambda[X] \to s_{\lambda'}[X]$. The involution ω also transforms the elementary symmetric functions to the homogeneous symmetric functions so $\tilde{S}_m$ is given by the formula

$$\tilde{S}_m = \sum_{i \geq 0} e_{m+i}[X] h_i^\perp (-1)^i$$

For $m \geq 0$ the formula can be given in plethystic notation as

$$\tilde{S}_m P[X] = (-1)^m P\left[X + \frac{1}{z}\right] \Omega[-zX]\Big|_{z^m}$$

The combinatorial action of $\tilde{S}_m$ on a Schur function is going to be the transpose of the rule given in proposition 2.4.1.

Before we leave this section we note that the vertex operator for the power symmetric function is just multiplication by m^{th} power symmetric function. This operation also has a well known combinatorial rule for computing the action on the Schur symmetric functions. The rule is called the Murnaghan-Nakayama rule (sometimes nicknamed the "slinky" rule). See [M] p. 48 for the combinatorial action, but we note that it has some similarities to the "snake" rule of proposition 2.4.1.

2.5 The Hall-Littlewood basis

The Hall-Littlewood symmetric functions $H_\mu[X;t]$ can be defined by the following formula.

Definition 2.5.1. *The Hall-Littlewood symmetric function*

$$H_\mu[X;t] = \Omega[XZ] \prod_{1 \le i \le j \le k} \frac{1 - z_j/z_i}{1 - tz_j/z_i}\Big|_{Z^\mu}$$

where μ is a partition.

These symmetric functions are not the same, but are related to the symmetric functions referred to as Hall-Littlewood polynomials in [M] p. 208. The Hall-Littlewood functions are related to the Schur symmetric functions by letting $t \to 0$ and to the homogeneous symmetric functions by letting $t \to 1$.

The Hall-Littlewood functions can be expanded in terms of the Schur symmetric function basis with coefficients $K_{\lambda\mu}(t)$, that is, $H_\lambda[X;t] = \sum_\lambda K_{\lambda\mu}(t) s_\mu[X]$. The $K_{\lambda\mu}(t)$ are well studied and referred to as the Kostka-Foulkes coefficients. They are known to be polynomials in t with positive integer coefficients [Fo].

$K_{\lambda\mu}(0) = 1$ if $\lambda = \mu$, and $K_{\lambda\mu}(0) = 0$ otherwise. $K_{\lambda\mu}(1) = K_{\lambda\mu} =$ the number of column strict tableau of shape λ and content μ. The numbers $K_{\lambda\mu}$ are referred to as the Kostka numbers. The $K_{\lambda\mu}(t)$ are a t counting of the column strict tableau of shape λ and content μ. It was conjectured by Foulkes that there should be a statistic such that

$$K_{\lambda\mu}(t) = \sum_{T \in CST_\lambda^\mu} t^{c(T)} \tag{2.10}$$

This conjecture was answered by Lascoux and Schützenberger [LS1] and the statistic on the column strict tableau that they introduced was the charge of the tableau.

Example 2.5.1. $\lambda = (3,2,1)$, $\mu = (2,2,1,1)$

$$\begin{array}{rccccccc}
 & \begin{array}{|c|c|c|}\hline 4 & \multicolumn{2}{l}{} \\ \hline 2 & 2 & \multicolumn{1}{l}{} \\ \hline 1 & 1 & 3 \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{l}{} \\ \hline 2 & 4 & \multicolumn{1}{l}{} \\ \hline 1 & 1 & 2 \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{l}{} \\ \hline 2 & 2 & \multicolumn{1}{l}{} \\ \hline 1 & 1 & 4 \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 4 & \multicolumn{2}{l}{} \\ \hline 2 & 3 & \multicolumn{1}{l}{} \\ \hline 1 & 1 & 2 \\ \hline \end{array} \\
charge = & 2 & & 2 & & 1 & & 3 \\
K_{(3,2,1)(2,2,1,1)}(t) = & t^2 & + & t^2 & + & t^1 & + & t^3
\end{array}$$

The vertex operator for the $H_\mu[X;t]$ is given by the following formula.

Definition 2.5.2. *The Hall-Littlewood vertex operator for $m \geq 0$*

$$H_m = \sum_{i,j\geq 0} h_{m+i+j}[X](-1)^i t^j e_i^{\perp} h_j^{\perp} = \sum_{j\geq 0} S_{m+j} h_j^{\perp} t^j$$

This definition may seem like it contains an infinite sum but notice that for sufficiently large values of i and j ($i+j > deg(P)$ is enough) $e_i^{\perp} h_j^{\perp} P[X] = 0$.

Note that the H_m operator also has a definition in terms of plethystic notation.

$$H_m P[X] = P\left[X - \frac{1-t}{z}\right] \Omega[zX]\Big|_{z^m} \tag{2.11}$$

The generating function for the Hall-Littlewood vertex operator will be denoted by $\mathbf{H}(z)$ and is given by its action on an arbitrary symmetric function $P[X]$ by

$$\mathbf{H}(z)P[X] = \sum_m z^m H_m P[X] \tag{2.12}$$

$$= P\left[X - \frac{1-t}{z}\right] \Omega[zX] \tag{2.13}$$

We state and give a proof of the vertex operator property of the H_m operator.

Proposition 2.5.1. *For $m \geq \mu_1$ we have that*

$$H_m H_\mu[X;t] = H_{(m,\mu)}[X;t]$$

where the notation (m,μ) represents the partition $(m, \mu_1, \ldots, \mu_k)$.

Proof. Using the definitions of $H_\mu[X;t]$ and H_m in a analogous manner to the Schur function vertex operator property we have that

$$\begin{aligned} H_m H_\mu[X;t] &= H_m \left(\prod_{1\leq i\leq j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i} \Omega[XZ]\Big|_{Z^\mu} \right) \\ &= \prod_{1\leq i\leq j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i} \Omega\left[\left(X - \frac{1-t}{z_o}\right) Z\right]\Big|_{Z^\mu} \Omega[z_o X]\Big|_{z_o^m} \\ &= \prod_{1\leq i\leq j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i} \Omega[XZ]\Omega[-Z/z_o]\Omega[tZ/z_o]\Omega[z_o X]\Big|_{Z^\mu}\Big|_{z_o^m} \end{aligned}$$

$$= \prod_{1\le i\le j\le k} \frac{1 - z_j/z_i}{1 - tz_j/z_i} \Omega[X(z_o + Z)] \prod_{j=1}^{k} \frac{1 - z_j/z_o}{1 - tz_j/z_o}\Big|_{Z^\mu}\Big|_{z_o^m}$$
$$= \prod_{0\le i\le j\le k} \frac{1 - z_j/z_i}{1 - tz_j/z_i} \Omega[X(z_o + Z)]\Big|_{z_o^m Z^\mu}$$
$$= H_{(m,\mu)}[X; t]$$

□

If $t \to 1$, then the vertex operator $H_m\Big|_{t\to 1}$ tends to multiplication by $h_m[X]$. If $t \to 0$, then the vertex operator $H_m\Big|_{t\to 0}$ tends to the Schur function vertex operator. The action of H_m on the Schur function basis is therefore a generalization of both the Pieri rule for multiplication by $h_m[X]$ and the Schur function vertex operator.

The H_m operator also has a commutation relation similar to the one for the Schur function vertex operator S_m. When the expression $\mathbf{H}(z_1)\mathbf{H}(z_2)$ is applied to an arbitrary symmetric function $P[X]$ we have that

$$\begin{aligned}
\mathbf{H}(z_1)\mathbf{H}(z_2)P[X] &= \mathbf{H}(z_1)P\left[X - \frac{1-t}{z_2}\right]\Omega[z_2X] \\
&= P\left[X - \frac{1-t}{z_1} - \frac{1-t}{z_2}\right]\Omega\left[z_2\left(X - \frac{1-t}{z_1}\right)\right]\Omega[z_1X] \\
&= P\left[X - \frac{1-t}{z_1} - \frac{1-t}{z_2}\right]\Omega[z_1X]\Omega[z_2X]\frac{1 - z_2/z_1}{1 - tz_2/z_1} \\
&= \frac{1 - z_2/z_1}{1 - tz_2/z_1}\frac{1 - tz_1/z_2}{1 - z_1/z_2}\mathbf{H}(z_2)\mathbf{H}(z_1)P[X] \\
&= \frac{tz_1 - z_2}{z_1 - tz_2}\mathbf{H}(z_2)\mathbf{H}(z_1)P[X]
\end{aligned}$$

Taking coefficients of $z_1^{m+1}z_2^n$ on both sides of the equation

$$(z_1 - tz_2)\mathbf{H}(z_1)\mathbf{H}(z_2)P[X] = (tz_1 - z_2)\mathbf{H}(z_2)\mathbf{H}(z_1)P[X]$$

gives the relation

$$H_mH_n - tH_{m+1}H_{n-1} = tH_nH_m - H_{n-1}H_{m+1} \tag{2.14}$$

In particular, when $n = m + 1$ we have $H_mH_{m+1} = tH_{m+1}H_m$.

2.6 The combinatorial interpretation of $H_\mu[X;t]$, $s_\lambda[X]$, and H_m

It has been mentioned already that by a theorem of Lascoux and Schützenberger, the $K_{\lambda\mu}(t)$ are a generating function for the column strict tableau of shape λ and content μ. These coefficients are polynomials with one term for each of the elements of CST_λ^μ and each term keeps track of the charge of a tableau.

Since the coefficients of the expansion of $H_\mu[X;t]$ in terms of the Schur functions are $K_{\lambda\mu}(t)$ then by equation (2.10) we have that

$$H_\mu[X;t] = \sum_{T\in CST^\mu} t^{c(T)} s_{\lambda(T)}[X] \tag{2.15}$$

where $c(T)$ is the charge of the column strict tableau and $\lambda(T)$ is the shape.

This equation suggests that the polynomials $H_\mu[X;t]$ can be thought of as a generating function for the column strict tableaux of content μ. The Schur functions in this equation represent a placeholder for the shape of the column strict tableau, and the t exponent keeps track of the charge.

This also gives an interpretation of the operator H_m since it has the property that $H_m H_\mu[X;t] = H_{(m,\mu)}[X;t]$, it is an operator that changes the generating function for column strict tableaux of content μ to the generating function for column strict tableaux of content (m,μ). The action of H_m on a single term $t^d s_\lambda[X]$ when expanded again in terms of Schur functions represents the effect on a single tableau of shape λ and charge d of raising the content by a row of length m.

The operator can be programmed fairly easily in Maple using the symmetric function package written by John Stembridge. The first observation to make when looking at H_m acting on the Schur functions is that the coefficients are all of the form t^a, $t^a - t^b$, or 0.

Example 2.6.1. *The action of H_1 on a Schur function $s_{(4,2,1)}[X]$*

$$\begin{aligned} H_1 s_{(4,2,1)}[X] =& t^4 s_{(5,2,1)} + t^3 s_{(4,3,1)} + t^3 s_{(4,2,2)} + t^3 s_{(4,2,1,1)} + \\ & \left(t^2 - t\right) s_{(3,3,2)} + \left(t^2 - t\right) s_{(3,3,1,1)} + \left(t^2 - 1\right) s_{(3,2,2,1)} \end{aligned}$$

The fact that there are negative coefficients in the expansion $H_m s_\lambda[X]$ is not the most ideal combinatorial situation. One would hope that the operation that raises the content of a tableau produces another set of tableaux and $H_m s_\lambda[X]$ would be the sum over the weights of those. We will show in the next chapter that the operator that acts on tableaux and raises the content from μ to (m, μ) can be realized to actually give a proof of equation (2.15).

2.7 The H_m operator on the Schur functions

It was noted that when $t \to 1$ that the operator $H_m \to$ multiplication by $h_m[X]$, and as $t \to 0$ the operator $H_m \to S_m$. The combinatorial action of H_m on the Schur functions will therefore be a generalization of the Pieri rule and of the "snake" rule of Proposition 2.4.1.

The following proposition describes the combinatorial rule for computing the action of H_m on the Schur functions.

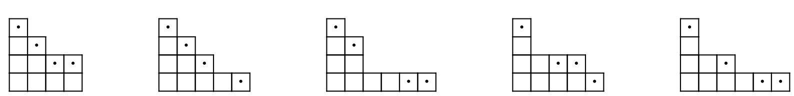
Proposition 2.7.1. *Let λ be a partition of n, let m be a non-negative integer and let k be any non-negative integer such that $m + k \geq \lambda_1$, then*

$$H_m s_\lambda[X] = \sum_{\mu/\lambda \in \mathcal{H}_{m+k}} (-1)^{ht_k(\mu)-1} t^{|\lambda/\mu^r|} s_{\mu\rfloor_k}[X]$$

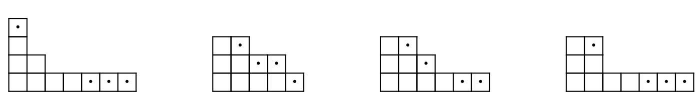
with the understanding that if $\mu\rfloor_k$ is not defined then there is no contribution from that term.

The LHS of Proposition 2.7.1 is independent of the choice of k, the RHS is also, but less obviously so.

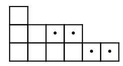
Example 2.7.1. *The action of H_1 on $s_{(4,2,1)}[X]$ using $k = 3$*



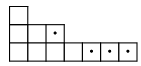
$$H_1 s_{(4,2,1)}[X] = -s_{(3,2,2,1)}[X] + 0 \quad + t^2 s_{(3,2,2,1)}[X] - t s_{(3,3,1,1)}[X] + t^2 s_{(3,3,1,1)}[X]$$



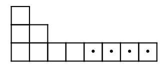
$$+t^3 s_{(4,2,1,1)}[X] - t s_{(3,3,2)}[X] + t^2 s_{(3,3,2)}[X] + t^3 s_{(4,2,2)}[X]$$



$+0$



$+ t^3 s_{(4,3,1)}[X]$



$+ t^4 s_{(5,2,1)}[X]$

Proof. Recall that one expression for the Hall-Littlewood vertex operator is

$$H_m = \sum_{i \geq 0} S_{m+i} h_i^{\perp} t^i$$

Using the result of Proposition 2.4.1 we have

$$H_m s_\lambda[X] = \sum_{i\geq 0} t^i S_{m+i} h_i^{\perp} s_\lambda[X] \tag{2.16}$$

$$= \sum_{i\geq 0} t^i \sum_{\rho:\lambda/\rho \in \mathcal{H}_i} S_{m+i} s_\rho[X] \tag{2.17}$$

$$= \sum_{i\geq 0} t^i \sum_{\rho:\lambda/\rho \in \mathcal{H}_i} s_{(m+i+k,\rho)\rfloor_k}[X](-1)^{ht_k((m+i+k,\rho))-1} \tag{2.18}$$

As always, the equation holds with the understanding that $s_{(m+i+k,\rho)\rfloor_k}[X] = 0$ if removing the k-snake does not leave a partition.

Since λ/ρ is a horizontal strip of size i, then $(m+k+i,\rho)/\lambda$ will be a horizontal strip also, but of size $m+k$. Therefore equation (2.18) becomes

$$H_m s_\lambda[X] = \sum_{i\geq 0} t^i \sum_{(m+i+k,\rho)/\lambda \in \mathcal{H}_{m+k}} s_{(m+i+k,\rho)\rfloor_k}[X](-1)^{ht_k((m+i+k,\rho))-1} \tag{2.19}$$

$$= \sum_{i\geq 0} t^i \sum_{\mu} s_{\mu\rfloor_k}[X](-1)^{ht_k(\mu)-1} \tag{2.20}$$

Where the last sum is over all μ such that $\mu/\lambda \in \mathcal{H}_{m+k}$ and $\mu_1 = m+k+i$. Since $|\mu| - |\lambda| = m+k$ then $|\lambda/\mu^r| = |\lambda| - |\mu^r| = |\lambda| - (|\mu| - (m+k+i)) = i$. Therefore equation (2.20) can be rewritten as

$$H_m s_\lambda[X] = \sum_{\mu/\lambda \in \mathcal{H}_{m+k}} t^{|\lambda/\mu^r|} s_{\mu\rfloor_k}[X](-1)^{ht_k(\mu)-1} \tag{2.21}$$

$\square$

2.8 The commutation relation of H_m and Γ_{1^k}

The operator $e_k^{\perp}$ is of interest because of it's relation with the operator H_m. To be consistent with other papers written on the subject we will rename the operator Γ_{1^k} (see [GP], [G]).

The action of Γ_{1^k} on the Schur functions is given by its Pieri formula definition 2.3.3. When Γ_{1^k} acts on a Schur function $s_\lambda[X]$ it produces a sum over all Schur functions indexed by partitions ρ that differ from λ by a vertical k-strip. The Schur function $s_\lambda[X]$ represents a place holder for the shape of a tableau in the formula (2.15). The operator Γ_{1^k} can be recognized as an operator that removes all possible vertical strips of size k from the shape of the tableau, but does not change the charge (since the t coefficient multiplied by the Schur function remains the same).

A generating function for this operator can be expressed plethystically by the formula

$$\Gamma(u)P[X] = \sum_{k\geq 0}(-u)^k\Gamma_{1^k}P[X] = P[X-u]$$

Notice that we have

$$\begin{aligned}
\Gamma(u)\mathbf{H}(z)P[X] &= \Gamma(u)P\left[X-\frac{1-t}{z}\right]\Omega[zX] \\
&= P\left[X-u-\frac{1-t}{z}\right]\Omega[z(X-u)] \\
&= (1-uz)\mathbf{H}(z)\Gamma(u)P[X]
\end{aligned}$$

By equating coefficients of $u^k z^m$ on both sides of this result we have that

$$\Gamma_{1^k}H_m = H_m\Gamma_{1^k} + H_{m-1}\Gamma_{1^{k-1}} \tag{2.22}$$

From this identity we derive the following formula

$$\Gamma_{1^k}H_\mu[X;t] = \sum_{I\subset\{1..l(\mu)\}} H_{\mu_1-\chi(1\in I)}H_{\mu_2-\chi(2\in I)}\cdots H_{\mu_{l(\mu)}-\chi(l(\mu)\in I)}1 \tag{2.23}$$

where the sum is over subsets I of size k and the function $\chi(B)$ is 1 if B is true and 0 if B is false. Recall that there is a simple commutation relation for H_m and H_{m+1}, namely $H_m H_{m+1} = tH_{m+1}H_m$, therefore the right hand side of the equation is the sum over Hall-Littlewood symmetric functions with t coefficients. The exponent for each t is found by the number of operations required to straighten $\mu - I$ to a partition.

Example 2.8.1. *The operator Γ_1 acting on the Hall-Littlewood symmetric function $H_{(2,2,2,1)}[X;t]$*

$$\begin{aligned}\Gamma_1 H_{(2,2,2,1)}[X;t] &= \Gamma_1 H_2 H_2 H_2 H_1 1\\ &= H_1 H_2 H_2 H_1 1 + H_2 H_1 H_2 H_1 1 + H_2 H_2 H_1 H_1 1 + H_2 H_2 H_2 H_0 1\\ &= (t^2+t+1)H_{(2,2,1,1)}[X;t] + H_{(2,2,2)}[X;t]\end{aligned}$$

Let $\mu^{(I)}$ be the tuple $(\mu_1 - \chi(1 \in I), \mu_2 - \chi(2 \in I) \cdots, \mu_{l(\mu)} - \chi(l(\mu) \in I))$ after it is rearranged so the result is a partition. Also we define $\tilde{K}_{\lambda\mu}(t) = K_{\lambda\mu}(1/t)t^{n(\mu)}$. Because $co(T) = n(\mu) - c(T)$, it must be that the $\tilde{K}_{\lambda\mu}(t)$ are also generating functions for the column strict tableau of of shape λ and content μ with cocharge as the statistic. That is,

$$\tilde{K}_{\lambda\mu}(t) = \sum_{T \in CST_\lambda^\mu} t^{co(T)}$$

Garsia and Procesi show in [GP] that by equating coefficients of $s_\rho[X]$ on both sides of equation (2.23) and then performing some simplification on the resulting formula then it reduces to

$$\sum_{\lambda/\rho \in \mathcal{V}_k} \tilde{K}_{\lambda\mu}(t) = \sum_{\substack{I \subset \{1,\ldots,l(\mu)\}\\ |I|=k}} t^{\sum_{i\in I}(i-1)} \tilde{K}_{\rho\mu^{(I)}}(t) \tag{2.24}$$

A bijective proof of this equation was the original motivation for the author's interest in this area of algebraic combinatorics.

Chapter 3

More Tableau Operators

3.1 The operator $\mathbf{H}_m^\rho$

Let T be a tableau of shape λ and let ρ be a partition such that $\rho/\lambda \in \mathcal{H}_{m+n}$ where $n = |\lambda|$. Define $\mathbf{H}_m^\rho T$ by the following procedure

1. Form a tableau by changing all the 1's in T to 2's, all the 2's in T to 3's, etc. Denote this type of change to the content by $T \uparrow$. Place the resulting tableau on a row of $m+n$ 1's to make a tableau, $\mathbf{S}_{m+n}T$, of shape $\lambda(\mathbf{S}_{m+n}T) = (m+n, \lambda(T))$ and content $\mu(\mathbf{S}_{m+n}T) = (m+n, \mu(T))$.

2. $\mathbf{S}_{m+n}T$ is transformed into a tableau of shape ρ by performing cyclage operations until it is of the correct shape. Consider the cells corresponding to $\lambda(\mathbf{S}_{m+n}T)^r/\rho^r$, if these are removed and placed in the first row of the shape then the tableau would be of shape ρ. Perform one cyclage operation for each one of these cells, starting from the right and working to the left. The bumping path of column evacuation of these cells will never cross and this guarantees that the cells will end up in the first row when they are inserted. Call the resulting tableau $T^\rho = \mathbf{H}_m^\rho T$.

$\mathbf{S}_{m+n}T$ will have charge equal to the charge of T. This follows by remarks made in the previous chapter.

The shape of $\lambda(T^\rho)$ will be ρ. The number of cyclage operations performed is $|\lambda(\mathbf{S}_{m+n}T)^r/\rho^r| = |\lambda/\rho^r|$ therefore $c(T^\rho) = c(\mathbf{S}_{m+n}T) + |\lambda/\rho^r| = c(T) + |\lambda/\rho^r|$.

Example 3.1.1. *Let $\rho = (11, 2, 2, 1)$. To calculated the action of $\mathbf{H}_2^\rho$ on the tableau* $T = \begin{array}{|c|c|c}\cline{1-2} 4 & 5 & \\ \cline{1-2} 2 & 3 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$ *first calculate the intermediate tableau*

$$\mathbf{S}_{m+n}T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\cline{1-2} 5 & 6 & \multicolumn{7}{c}{} \\ \cline{1-2} 3 & 4 & \multicolumn{7}{c}{} \\ \cline{1-3} 2 & 2 & 3 & \multicolumn{6}{c}{} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}$$

we need to perform a cyclage two times on $\mathbf{S}_{m+n}T$ to obtain a tableau of shape ρ. $\begin{array}{|c|c|c|c|c|c|c|c|c|}\cline{1-2} & \times & \multicolumn{7}{c}{} \\ \cline{1-2} & & \multicolumn{7}{c}{} \\ \cline{1-3} & & \times & \multicolumn{6}{c}{} \\ \hline & & & & & & & & \\ \hline\end{array}$ . . *Then the resulting tableau is*

$$\mathbf{H}_2^{(11,2,2,1)}T = T^\rho = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\cline{1-1} 6 & \multicolumn{10}{c}{} \\ \cline{1-2} 3 & 4 & \multicolumn{9}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{9}{c}{} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 5 \\ \hline\end{array}$$

The charge of T and $\mathbf{S}_{m+n}T$ is 4, the charge of $T^\rho = 6$.

Example 3.1.2. *Let $\rho = (10, 3, 1)$. Calculate the action of $\mathbf{H}_2^\rho$ on the tableau* $T = \begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 3 \\ \hline\end{array}$. *Create a tableau $\mathbf{S}_{m+n}T$ of content $(8, 2, 2, 1, 1)$ and shape $(8, 3, 2, 1)$ as an intermediate step, the charge of $\mathbf{S}_{m+n}T = 2$ is the same as the charge of T.*

$$\mathbf{S}_{m+n}T = \begin{array}{|c|c|c|c|c|c|c|c|}\cline{1-1} 5 & \multicolumn{7}{c}{} \\ \cline{1-2} 3 & 3 & \multicolumn{6}{c}{} \\ \cline{1-3} 2 & 2 & 4 & \multicolumn{5}{c}{} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}$$

Two cyclage operations must be done on $\mathbf{S}_{m+n}T$ to obtain

$$\mathbf{H}_2^{(10,3,1)}T = T^\rho = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-1} 3 & \multicolumn{9}{c}{} \\ \cline{1-3} 2 & 2 & 4 & \multicolumn{7}{c}{} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 5 \\ \hline\end{array}$$

The charge of T^ρ is 4.

$\mathbf{H}_m^\rho$ is an invertible operator. Given any tableau, S, of content $(m+n, \mu)$, it is possible to determine a tableau, T,of content μ such that $\mathbf{H}_m^{\lambda(S)}T = S$, by reversing the steps in this process.

To recover T from S, first recognize all of the cells in the first row that are not 1 are cells to be uncyclaged. Perform one uncyclage operation on the first corner for every cell in the first row that is not labeled with a 1. After these uncyclage operations, the result will be a tableau with the first row consisting only of 1's. The tableau T is found by discarding the row of 1's and changing 2's to 1's, 3's to 2's, etc. in the remainder.

Example 3.1.3. *The tableau*

$$S = \begin{ytableau} 3 & 4 \\ 2 & 2 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 \end{ytableau}$$

may be obtained from an application of $\mathbf{H}_1^{(9,4,2)}$ *to some tableau* T*. The* $\boxed{5}$ *in the first row must have been placed there from a cyclage operation, so uncyclaging the* $\boxed{5}$ *yields the tableau*

$$\begin{ytableau} 5 \\ 3 & 4 \\ 2 & 2 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{ytableau}$$

and this is the $\mathbf{S}_{m+n}T$ *that corresponds to the tableau*

$$T = \begin{ytableau} 4 \\ 2 & 3 \\ 1 & 1 & 1 & 5 \end{ytableau}$$

therefore $\mathbf{H}_1^{\rho}T = S$*.*

Example 3.1.4. *Let* $S = \begin{ytableau} 3 & 4 \\ 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 5 \end{ytableau}$ *be a tableau of content* $\mu(S) = (10, 4, 2, 1)$ *and of shape* $\lambda(S) = (12, 4, 2)$*.* S *is in the image of the operator* $\mathbf{H}_2^{(12,4,2)}$ *thus there is some tableau* T *such that* $\mathbf{H}_2^{(12,4,2)}T = S$*.* $\mathbf{S}_{m+n}T$ *can be found by uncyclaging the* $\boxed{2}$ *and the* $\boxed{5}$ *from the first row to obtain*

$$\mathbf{S}_{m+n}T = \begin{ytableau} 5 \\ 3 & 4 \\ 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{ytableau}$$

then T *must be the tableau*

$$T = \begin{ytableau} 4 \\ 2 & 3 \\ 1 & 1 & 1 & 1 & 2 \end{ytableau}$$

Define the operator $\mathbf{H}_m$ to be an operator on a single tableau T that produces a set of tableaux of content $(m+n,\mu)$.

We let

$$\mathbf{H}_m T = \{\mathbf{H}_m^{\rho} T\}_{\rho/\lambda(T)\in\mathcal{H}_{m+n}}$$

Example 3.1.5. *The operator $\mathbf{H}_2$ acting on all tableau of content (1^3), notice that all tableau of content $(5,1^3)$ appear on the RHS exactly once.*

$$\mathbf{H}_2 \begin{ytableau} 1 & 2 & 3 \end{ytableau} = \left\{ \begin{ytableau} 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{ytableau},\ \begin{ytableau} 3 & 4 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{ytableau},\ \begin{ytableau} 4 \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 \end{ytableau},\ \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 \end{ytableau} \right\}$$

$$\mathbf{H}_2 \begin{ytableau} 3 \\ 1 & 2 \end{ytableau} = \left\{ \begin{ytableau} 4 \\ 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{ytableau},\ \begin{ytableau} 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{ytableau},\ \begin{ytableau} 4 \\ 3 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{ytableau},\ \begin{ytableau} 3 \\ 1 & 1 & 1 & 1 & 1 & 2 & 4 \end{ytableau} \right\}$$

$$\mathbf{H}_2 \begin{ytableau} 2 \\ 1 & 3 \end{ytableau} = \left\{ \begin{ytableau} 3 \\ 2 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{ytableau},\ \begin{ytableau} 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{ytableau},\ \begin{ytableau} 4 \\ 2 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{ytableau},\ \begin{ytableau} 2 \\ 1 & 1 & 1 & 1 & 1 & 3 & 4 \end{ytableau} \right\}$$

$$\mathbf{H}_2 \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau} = \left\{ \begin{ytableau} 4 \\ 3 \\ 2 \\ 1 & 1 & 1 & 1 & 1 \end{ytableau},\ \begin{ytableau} 3 \\ 2 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{ytableau} \right\}$$

3.2 A formula for $H_{(m,\mu)}$

The definition of $\mathbf{H}_m$ was chosen to mimic on tableaux the action that H_m has on the Schur function basis. Define a weight function for the tableaux of content $(m+n,\mu)$. $W_n(T)=0$ if $\lambda(T)$ does not have a n-snake and $W_n(T)=t^{c(T)}(-1)^{ht_n(\lambda(T))-1}$ $s_{\lambda(T)\rfloor_n}[X]$ otherwise. By Proposition 2.7.1 we have that $H_m t^{c(T)} s_{\lambda(T)}[X]$ is given by

$$\begin{aligned} H_m t^{c(T)} s_{\lambda(T)}[X] &= \sum_{\substack{\rho/\lambda(T)\in\mathcal{H}_{m+n} \\ \rho\rfloor_n exists}} t^{c(T)+|\lambda(T)/\rho^r|}(-1)^{ht_n(\rho)-1} s_{\rho\rfloor_n}[X] \\ &= \sum_{\substack{S\in\mathbf{H}_m T \\ \lambda(S)\rfloor_n exists}} t^{c(S)}(-1)^{ht_n(\lambda(S))-1} s_{\lambda(S)\rfloor_n}[X] \\ &= \sum_{S\in\mathbf{H}_m T} W_n(S) \end{aligned}$$

By the theorem of Lascoux and Schützenberger, we know that the polynomials $H_\mu[X;t]$ are a generating function for all of the tableau of content μ, that is they satisfy

$$H_\mu[X;t] = \sum_{T \in CST^\mu} t^{c(T)} s_{\lambda(T)}[X] \tag{3.1}$$

Just by noting that H_m is an invertible operator we have also that

Theorem 3.2.1. *Let $\mu \vdash n$.*
If $H_\mu[X;t] = \sum_{T \in CST^\mu} t^{c(T)} s_{\lambda(T)}[X]$, then

$$H_m H_\mu[X;t] = \sum_{T \in CST^{(m+n,\mu)}} W_n(T)$$

Example 3.2.1. *$H_{(1^3)}[X;t]$ is the generating function for the tableaux of content (1^3)*

$$\begin{array}{lll} 1 & 2 & 3 \end{array} \qquad \begin{array}{ll} 3 & \\ 1 & 2 \end{array} \qquad \begin{array}{ll} 2 & \\ 1 & 3 \end{array} \qquad \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

$$H_{(1^3)}[X;t] = t^3 s_{(3)}[X] + t^2 s_{(2,1)}[X] + t s_{(2,1)}[X] + s_{(1^3)}[X]$$

The image of all of the tableaux of content (1^3) is the set of tableaux of content $(5,1^3)$.

$$\begin{array}{lllll} 2 & 3 & 4 & & \\ 1 & 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{llllll} 3 & 4 & & & & \\ 1 & 1 & 1 & 1 & 1 & 2 \end{array} \quad \begin{array}{lllllll} 4 & & & & & & \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 \end{array} \quad \begin{array}{llllllll} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 \end{array}$$

$$\begin{array}{lllll} 4 & & & & \\ 2 & 3 & & & \\ 1 & 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{llllll} 2 & 3 & & & & \\ 1 & 1 & 1 & 1 & 1 & 4 \end{array} \quad \begin{array}{llllll} 4 & & & & & \\ 3 & & & & & \\ 1 & 1 & 1 & 1 & 1 & 2 \end{array} \quad \begin{array}{lllllll} 3 & & & & & & \\ 1 & 1 & 1 & 1 & 1 & 2 & 4 \end{array}$$

$$\begin{array}{lllll} 3 & & & & \\ 2 & 4 & & & \\ 1 & 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{llllll} 2 & 4 & & & & \\ 1 & 1 & 1 & 1 & 1 & 3 \end{array} \quad \begin{array}{llllll} 4 & & & & & \\ 2 & & & & & \\ 1 & 1 & 1 & 1 & 1 & 3 \end{array} \quad \begin{array}{lllllll} 2 & & & & & & \\ 1 & 1 & 1 & 1 & 1 & 3 & 4 \end{array}$$

$$\begin{array}{lllll} 4 & & & & \\ 3 & & & & \\ 2 & & & & \\ 1 & 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{llllll} 3 & & & & & \\ 2 & & & & & \\ 1 & 1 & 1 & 1 & 1 & 4 \end{array}$$

When each of these tableau are counted with weight W_3, we have an expression for the symmetric function $H_1H_{(1^3)}[X;t] = H_{(1^4)}[X;t]$.

$$\begin{aligned} H_1H_{(1^3)}[X;t] = &- t^3 s_{(2,2)}[X] + t^4 s_{(2,2)}[X] + t^5 s_{(3,1)}[X] + t^6 s_{(4)}[X] \\ &+ 0 + t^3 s_{(2,2)}[X] + t^3 s_{(2,1,1)}[X] + t^4 s_{(3,1)}[X] \\ &+ 0 + t^2 s_{(2,2)}[X] + t^2 s_{(2,1,1)}[X] + t^3 s_{(3,1)}[X] \\ &+ t^0 s_{(1,1,1,1)}[X] + t^1 s_{(2,1,1)}[X] \end{aligned}$$

3.3 Implication by induction that $H_{(m,\mu)}[X;t]$ is a generating function

Theorem 3.2.1 says that $H_{(m,\mu)}[X;t]$ is a weighted sum over column strict tableaux of content $(m+n,\mu)$ with both positive and negative terms. By equation (3.1) we also know that it is a weighted sum over column strict tableaux of content (m,μ) with only positive terms. It should be that there is a direct correspondence with the non-canceling terms of 3.2.1 with the column strict tableau of content (m,μ) and a sign reversing involution on the remaining terms.

Let μ be a partition of n, let m be an integer larger than or equal to μ_1, and let T be a tableau of content of $(m+n,\mu)$. Then we shall say that T is naturally isomorphic to $\tilde{T}$, a tableau of content (m,μ), if the operation of deleting the first n 1's in the first row of T and sliding only the cells in the first row to the left yields $\tilde{T}$. Because removing the n cells from the first row of $\lambda(T)$ yields a partition then $\lambda(\tilde{T}) = \lambda(T)\rfloor_n$ and the height of the n-snake is 1 therefore

$$W_n(T) = W_0(\tilde{T}) = t^{c(\tilde{T})} s_{\lambda(\tilde{T})}[X] \tag{3.2}$$

.

Example 3.3.1. *The tableau* $T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \cline{1-2} 3 & 5 \\ \cline{1-3} 2 & 2 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 \\ \hline \end{array}$ *is naturally isomorphic to the tableau* $\tilde{T} = \begin{array}{|c|c|c|c|} \cline{1-2} 3 & 5 \\ \cline{1-3} 2 & 2 & 4 \\ \hline 1 & 1 & 1 & 6 \\ \hline \end{array}$ *. The charge of both of these tableaux is 4.*

Example 3.3.2. *The tableau* $T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \cline{1-2} 3 & 5 \\ \cline{1-3} 2 & 2 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 \\ \hline \end{array}$ *is not naturally isomorphic to a tableau of content* $(2, 2, 1, 1, 1, 1)$ *because deleting* 6 *of the* 1*'s and bringing the tableau to straight shape changes more than just the cells in the first row.*

In general, there are some tableaux of content $(m+n, \mu)$ that will not correspond to a tableau of content (m, μ) ”naturally” (that is, it will be necessary to do more than just slide the cells of the bottom row to the left). Recall that the operator $K_n(T)$ was introduced to denote the tableau produced by deleting the first n cells labeled with 1s of a tableau T of content $(m + n, \mu)$ and then playing Jeu de Taquin to bring the tableau to straight shape. Recall also that K_n does not change the charge if $m \geq \mu_1$.

Let T be a tableau of content $(m + n, \mu)$. Assume that T is not naturally isomorphic to a tableau of content (m, μ) and also assume that $\lambda(T)$ has an n-snake. The claim is that there is an involution I_n producing a tableau $S = I_n(T)$ of content $(m+n, \mu)$ such that $K_n(S) = K_n(T)$ and $\lambda(S)\rfloor_n = \lambda(T)\rfloor_n$ and $ht_n(\lambda(S)) = ht_n(\lambda(T)) \pm 1$ so that $W_n(S) = -W_n(T)$.

Proof. (of claim) Let T be a column strict tableau of content $(m + n, \mu)$ that is not naturally isomorphic to a tableau of content (m, μ) and such that $\lambda(T)$ has an n-snake. Let $\rho = \lambda(K_n(T))$ and $\lambda = \lambda(T)\rfloor_n$ and $h = ht_n(\lambda(T))$. Define $\tilde{\lambda}$ to be the partition obtained by adding an n-snake of height $h+1$ to λ if $\rho_h > \lambda_h$ and the partition obtained by adding an n-snake of height $h - 1$ to λ if $\rho_h \leq \lambda_h$.

The image of the involution $I_n(T)$ is then defined to be the tableau of content $(m + n, \mu)$ of shape $\tilde{\lambda}$ such that $K_n(I_n(T)) = K_n(T)$. That is, let $I_n(T) = A_n^{\tilde{\lambda}}(K_n(T))$ ($A_n^{\tilde{\lambda}}(K_n(T))$ will exist because the leftmost cell of $\tilde{\lambda}/\lambda(K_n(T))$ is the same as the leftmost cell of $\lambda(T)/\lambda(K_n(T))$, since $A_n^{\lambda(T)}(K_n(T))$ exists then so must $A_n^{\tilde{\lambda}}(K_n(T))$).

If $\rho_h > \lambda_h$ then $\rho_{h+1} = \lambda(K_n(T))_{h+1} \leq \lambda(T)_{h+1} \leq \lambda_{h+1}$ so that $\rho_{h+1} \leq \lambda_{h+1}$. Also if $\rho_h \leq \lambda_h$ then $\rho_{h-1} = \lambda(K_n(T))_{h-1} \geq \lambda(T)_h > \lambda(T)_h - 1 = \lambda_{h-1}$ so that $\rho_{h-1} > \lambda_{h-1}$. These two statements together show that I_n is an involution.

I_n is not defined if $h = 1$ and $\rho_1 \leq \lambda_1$, but in this case $\rho = \lambda$ and T is naturally isomorphic to $K_n(T)$. $\square$

Example 3.3.3. *Consider the tableau* $T = \begin{array}{|c|c|c|c|}\cline{1-3} 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$ *that is an element of the image of* $\mathbf{H}_1$ *acting on the tableau of content* $(1,1,1)$*.* T *is not naturally isomorphic to a tableau of content* (1^4)*. Instead we have* $K_3(T) = \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 & 4 \\ \hline \end{array}$ *and* $\rho = (3,1)$*.* $W_3(T) = -t^3 s_{(2,2)}[X]$ *and* $\lambda = \lambda(T)\rfloor_3 = (2,2)$*. Because* $\rho_2 < \lambda_2$ *then* $\tilde{\lambda} = \lambda +$ *a 3-snake of height* 1*. There will be a corresponding tableau of shape* $\tilde{\lambda} = (5,2)$*. The tableau is found by computing* $I_3(T) = A_3^{(5,2)}(K_3(T)) = \begin{array}{|c|c|c|c|c|}\cline{1-2} 2 & 3 \\ \hline 1 & 1 & 1 & 1 & 4 \\ \hline \end{array}$*. The weight of* $I_3(T)$ *is of opposite sign of the weight as* T *since the height of the* 3*-snake of* $I_3(T)$ *is one less than the height of the* 3*-snake of* T*.* $W_3(I_3(T)) = t^3 s_{(2,2)}[X]$*.*

The existence of this involution on the tableaux of content $(m+n,\mu)$ that have an n-snake and are not naturally isomorphic to a tableaux of content (m,μ) shows that the sum

$$\sum_{T \in CST^{(m+n,\mu)}} W_n(T) = \sum_{T \in CST^{(m,\mu)}} W_0(T) \tag{3.3}$$

The proof of (3.1) now follows by induction on the number of parts of μ. For assume that

$$H_\mu[X;t] = \sum_{T \in CST^\mu} t^{c(T)} s_{\lambda(T)}[X]$$

for all partitions μ with less than or equal to k parts, then (m,μ) (with $m \geq \mu_1$) is a partition with $k+1$ parts. By Theorem 3.2.1 and then applying (3.3) we have that

$$\begin{aligned} H_{(m,\mu)}[X;t] &= \sum_{T \in CST^{(m+n,\mu)}} W_n(T) \\ &= \sum_{T \in CST^{(m,\mu)}} W_0(T) \\ &= \sum_{T \in CST^{(m,\mu)}} t^{c(T)} s_{\lambda(T)}[X] \end{aligned}$$

Example 3.3.4. *The tableaux of content* $(4,1,1,1)$ *fall into three categories. Two have the property that* $W_3(T) = 0$ *because* $\lambda(T)$ *does not have a* 3*-snake:*

$$\begin{array}{|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Two more have the property that they are images of each other under the involution $I_3(T)$ *and so have weight that is the opposite sign:*

$$\begin{array}{l} 2\,3\,4 \\ 1\,1\,1\,1 \end{array} \longleftrightarrow \begin{array}{l} 2\,3 \\ 1\,1\,1\,1\,4 \end{array}$$

The remaining 10 *tableaux are naturally isomorphic to tableaux of content* (1^4)

$$\begin{array}{l} 4 \\ 3 \\ 2 \\ 1\,1\,1\,1 \end{array} \longleftrightarrow \begin{array}{l} 4 \\ 3 \\ 2 \\ 1 \end{array} \qquad \begin{array}{l} 3 \\ 2 \\ 1\,1\,1\,1\,4 \end{array} \longleftrightarrow \begin{array}{l} 3 \\ 2 \\ 1\,4 \end{array}$$

$$\begin{array}{l} 4 \\ 2 \\ 1\,1\,1\,1\,3 \end{array} \longleftrightarrow \begin{array}{l} 4 \\ 2 \\ 1\,3 \end{array} \qquad \begin{array}{l} 4 \\ 3 \\ 1\,1\,1\,1\,2 \end{array} \longleftrightarrow \begin{array}{l} 4 \\ 3 \\ 1\,2 \end{array}$$

$$\begin{array}{l} 3\,4 \\ 1\,1\,1\,1\,2 \end{array} \longleftrightarrow \begin{array}{l} 3\,4 \\ 1\,2 \end{array} \qquad \begin{array}{l} 2\,4 \\ 1\,1\,1\,1\,3 \end{array} \longleftrightarrow \begin{array}{l} 2\,4 \\ 1\,3 \end{array}$$

$$\begin{array}{l} 4 \\ 1\,1\,1\,1\,2\,3 \end{array} \longleftrightarrow \begin{array}{l} 4 \\ 1\,2\,3 \end{array} \qquad \begin{array}{l} 3 \\ 1\,1\,1\,1\,2\,4 \end{array} \longleftrightarrow \begin{array}{l} 3 \\ 1\,2\,4 \end{array}$$

$$\begin{array}{l} 2 \\ 1\,1\,1\,1\,3\,4 \end{array} \longleftrightarrow \begin{array}{l} 2 \\ 1\,3\,4 \end{array} \qquad \begin{array}{l} 1\,1\,1\,1\,2\,3\,4 \end{array} \longleftrightarrow \begin{array}{l} 1\,2\,3\,4 \end{array}$$

3.4 Building tableaux of non-partition content

If $m < \mu_1$ the proof that $H_m H_\mu[X; q, t]$ is a generating function for the column strict tableaux of content (m, μ) does not work. The proof fails because if T is a tableau of content $(m + |\mu|, \mu)$ (where μ may be a composition and not just a partition), then deleting the $|\mu|$ 1s to create a tableau of content (m, μ) will change the charge if $m < \mu_i$. Always in the proof in the previous section we assumed that $m \geq \mu_1$ and so deleting $|\mu|$ 1s from T does not change the charge and the involution has the property that $c(T) = c(I_n(T))$.

Example 3.4.1. $H_1 H_3 1 = t^3 s_{(4)}[X] + t^2 s_{(3,1)}[X] + (t-1) s_{(2,2)}[X]$

$$\mathbf{H}_1 \begin{array}{l} 1\,1\,1 \end{array} = \begin{array}{l} 2\,2\,2 \\ 1\,1\,1\,1 \end{array} + \begin{array}{l} 2\,2 \\ 1\,1\,1\,1\,2 \end{array} + \begin{array}{l} 2 \\ 1\,1\,1\,1\,2\,2 \end{array} + \begin{array}{l} 1\,1\,1\,1\,2\,2\,2 \end{array}$$

charge 0 0 1 2 3

In the example above, each tableau is counted with weight $W_3(T)$*. The first two tableaux in* $\mathbf{H}_1\boxed{1\,1\,1}$ *have the same image in the map* K_3 *and they are images of each other in* *the involution I_3. Let $T_1 = \begin{array}{|c|c|c|c|}\hline 2&2&2 \\ \hline 1&1&1&1 \\ \hline\end{array}$ and $T_2 = \begin{array}{|c|c|c|c|c|}\hline 2&2 \\ \hline 1&1&1&1&2 \\ \hline\end{array}$, then the charge of $K_3(T_1) = K_3(T_2)$ does not equal the charge of either T_1 or T_2 and $c(T_1) = 0$ and $c(T_2) = 1$ the involution I_3 does not have the property that $c(T) = c(I_3(T))$.*

In the previous chapter it was shown that

$$H_m H_n - tH_{m+1}H_{n-1} = tH_n H_m - H_{n-1}H_{m+1} \tag{3.4}$$

and in particular, when $n = m + 1$ we have $H_m H_{m+1} = tH_{m+1}H_m$. This implies that if p is a composition with the property that if $i < j$ then $p_i \geq p_j - 1$ then the symmetric function $H_{p_1}H_{p_2}\cdots H_{p_l}1$ has positive coefficients when expanded in the Schur function basis. In fact, it is a power of t times the Hall-Littlewood symmetric function indexed by the rearrangement of the composition to a partition. In this section we would like to show that this symmetric function is a generating function for the tableau of content p.

Example 3.4.2. *Consider the symmetric function $H_2H_3H_11 = t^5 s_{(6)}[X] + \left(t^3 + t^4\right) s_{(5,1)}[X] + \left(t^2 + t^3\right) s_{(4,2)}[X] + t^2 s_{(4,1,1)}[X] + t^2 s_{(3,3)}[X] + ts_{(3,2,1)}[X] = tH_{(3,2,1)}[X;t]$*

$$\begin{array}{ccccc}
 & \begin{array}{cccccc} 1&1&2&2&2&3 \end{array} & \begin{array}{ccccc} 2 \\ 1&1&2&2&3 \end{array} & \begin{array}{ccccc} 3 \\ 1&1&2&2&2 \end{array} & \begin{array}{cccc} 2&3 \\ 1&1&2&2 \end{array} \\
\textit{charge=} & 5 & 4 & 3 & 3
\end{array}$$

$$\begin{array}{ccccc}
 & \begin{array}{cccc} 2&2 \\ 1&1&2&3 \end{array} & \begin{array}{cccc} 3 \\ 2 \\ 1&1&2&2 \end{array} & \begin{array}{ccc} 2&2&3 \\ 1&1&2 \end{array} & \begin{array}{ccc} 3 \\ 2&2 \\ 1&1&2 \end{array} \\
\textit{charge=} & 2 & 2 & 2 & 1
\end{array}$$

Let μ be a partition and so $H_\mu[X;t]$ is the generating function for all tableau of content μ and let m be such that $m \geq \mu_1 - 1$. We would like to show that $H_m H_\mu[X;t]$ is the generating function for the tableau for content (m, μ) and the correspondence defined in the previous section is the same. This statement has been shown in the previous section if $m \geq \mu_1$ but there is something to check if $m = \mu_1 - 1$ (if $m < \mu_1 - 1$ then the statement isn't true).

The tableau operator $\mathbf{H}_{\mu_1-1}$ acting on all tableaux of content μ produces all tableaux of content $(\mu_1 - 1 + |\mu|, \mu)$ and the symmetric function

$$H_{\mu_1-1}H_\mu[X;t] = \sum_{T \in CST^{(\mu_1-1+|\mu|,\mu)}} W_{|\mu|}(T)$$

is a generating function for these tableaux (where $W_n(T)$ is the weight function given earlier in this chapter).

Let T be a tableau of content $(\mu_1 - 1 + |\mu|, \mu)$ such that $W_{|\mu|}(T)$ is non-zero and T does not correspond to a tableau of content $(\mu_1 - 1, \mu)$ naturally. The involution $I_{|\mu|}$ exists but we need to show that $c(I_{|\mu|}(T)) = c(T)$. This happens because $K_{|\mu|-1}(T)$ is a tableau of partition weight and has the same charge as T and $I_{|\mu|-1}(T) = I_{|\mu|}(T)$. Therefore $c(T) = c(I_{|\mu|}(T))$ and so all tableaux of this type cancel.

If T is a tableau of content $(\mu_1 - 1 + |\mu|, \mu)$ that does correspond to a tableau $K_{|\mu|}(T)$ of content $(\mu_1 - 1, \mu)$ naturally, then $c(T) = c(K_{|\mu|}(T))$. The proof of this statement is not quite as obvious as it is for tableaux of partition content.

Proof. A cell labeled by 2 lies in the first row of T, otherwise T does not correspond to $K_{|\mu|}(T)$ naturally. Let $\tilde{T}$ be the tableau with the first cell labeled by a 2 changed to a 1 and then delete $|\mu|$ 1s. This tableau will have charge $c(T) - 1$ since deleting the extra 1s does not change the charge as long as there are more 1s than any other label and the operation of changing a 2 to a 1 commutes with cyclage so it does not change the cocharge and $n(\mu(\tilde{T})) = n(\mu(T)) - 1$.

Notice also that $\tilde{T} = (1,2)K_{|\mu|}(T)$ and since $c((i,i+1)T) = c(T) + \mu_i - \mu_{i+1}$ we have that $c(\tilde{T}) = c(K_{|\mu|}(T)) - 1$. Therefore $c(T) = c(K_{|\mu|}(T))$. □

Let p be any composition such that if $i < j$ then $p_i \geq p_j - 1$ and m be a non-negative integer such that $m \geq p_i - 1$ for all i. Let p^* be the partition formed by reordering the entries of the composition p.

Assume that by induction that $H_{p_1}H_{p_2}\cdots H_{p_l}1 = \sum t^{c(T)} s_{\lambda(T)}[X]$ where the sum is over all column strict tableaux of content p, then $H_m H_{p_1} H_{p_2} \cdots H_{p_l} 1$ is a generating function for the column strict tableaux of content $(m + |p|, p)$.

There is a 1-1 correspondence between $CST^{(m+|p|,p)}$ and $CST^{(m+|p|,p^*)}$ by applying transpositions $(i, i+1)$ where $i > 1$ and the difference between the charge of a tableau and its image under these transpositions is always the same, say $k = n(m,p) - n(m,p^*)$ (the cocharge stays fixed under transpositions $(i, i+1)$). This shows that

$$H_m H_{p_1} H_{p_2} \cdots H_{p_l} 1 = \sum_{CST^{(m+|p|,p)}} W_{|p|}(T) = \sum_{CST^{(m+|p|,p^*)}} t^k W_{|p|}(T)$$

By the previous argument we have that this is equal to

$$= \sum_{CST^{(m,p^*)}} t^k t^{c(T)} s_{\lambda(T)}[X]$$

Again, a sequence of transpositions of the form $(i, i+1)$ where $i > 1$ gives a correspondence between $CST^{(m,p^*)}$ and $CST^{(m,p)}$ and increases the charge of each tableau by exactly k so we can say that

$$H_m H_{p_1} H_{p_2} \cdots H_{p_l} 1 = \sum_{CST^{(m,p)}} t^{c(T)} s_{\lambda(T)}[X]$$

3.5 Unbuilding tableaux

Since when $m \geq \mu_1 - 1$ the tableaux of content (m, μ) are 'built' from the tableau of content μ by adding m 1's, this process can be reversed and can be used to calculate the charge of a tableau.

Any tableau, T of content μ and shape λ corresponds to a tableau, $\tilde{T}$ of content $(\mu_1 + |\mu^r|, \mu^r)$ and shape $(\lambda_1 + |\mu^r|, \lambda^r)$ found by sliding the cells of the first row of the tableau to the right and inserting $|\mu^r|$ 1s. Cast in terms of the operators that have already been described $\tilde{T} = A_{|\mu^r|}^{(\lambda_1+|\mu^r|,\lambda^r)}(T)$ and $T = K_{|\mu^r|}(\tilde{T})$.

$\tilde{T}$ can be recognized as the operator $\mathbf{H}_{\mu_1}^{(\lambda_1+|\mu^r|,\lambda^r)}$ applied to a tableau, S, of content μ^r. S is found by un-cyclaging the cells in the first row that are not labeled by 1 to create a tableau that can be recognized as $\mathbf{S}_{\mu_1}^{|\mu^r|}S$. Therefore we have that $T = K_{|\mu^r|}\mathbf{H}_m^{(\lambda_1+|\mu^r|,\lambda^r)}S$. By abuse of notation we will say that $T = \mathbf{H}_m^{\lambda}S$.

This gives an inductive method for calculating the charge of a tableau. Since the number of cells uncylaged in this process is equal to the number of cells that are not labeled by a 1 in the first row we will have that the charge is $c(T) = c(S) + \lambda_1 - \mu_1$.

By iterating this operation we have that (again, by abusing notation)

$$T = \mathbf{H}_{\mu_1}^{\nu^{(1)}} \mathbf{H}_{\mu_2}^{\nu^{(2)}} \cdots \mathbf{H}_{\mu_{l(\mu)}}^{\nu^{(l(\mu))}}.$$

where $|\nu^{(i)}| = \mu_i + \mu_{i+1} + \cdots + \mu_{l(\mu)}$, and $\nu^{(1)} = \lambda(T)$. The charge of this tableau will be the number of cells cyclaged in this process. In the i^{th} step the number cells cyclaged is $\nu^{(i)}{}_1 - \mu_i$. Therefore the charge of the tableau is given by $c(T) = \sum_{i=1}^{l(\mu)}(\nu^{(i)}{}_1 - \mu_i) = (\sum_{i=1}^{l(\mu)} \nu^{(i)}{}_1) - |\mu|$.

Example 3.5.1. *Consider the tableau*

$$T = \begin{array}{ccccc} 6&7&&& \\ 5&6&8&& \\ 3&4&7&& \\ 2&3&4&& \\ 1&1&2&5&9 \end{array}$$

$$\begin{array}{cccccccc} 6&7&&&&&& \\ 5&6&8&&&&& \\ 3&4&7&&&&& \\ 2&3&4&&&&& \\ &&&1&1&2&5&9 \end{array} \longrightarrow \begin{array}{ccccccc} 2&5&9&&&& \\ &&&6&7&& \\ &&&5&6&8& \\ &&&3&4&7& \\ &&&2&3&4& \\ &&&&&&1\;1 \end{array} \longrightarrow \begin{array}{ccccc} 9&&&& \\ 6&7&&& \\ 5&5&8&& \\ 3&4&6&& \\ 2&2&3&4&7 \end{array} \longrightarrow \begin{array}{ccccc} 8&&&& \\ 5&6&&& \\ 4&4&7&& \\ 2&3&5&& \\ 1&1&2&3&6 \end{array}$$

$$S = \begin{array}{ccccc} 8&&&& \\ 5&6&&& \\ 4&4&7&& \\ 2&3&5&& \\ 1&1&2&3&6 \end{array}$$

$$c(T) = c(S) + 3$$

$$T = \mathbf{H}_2^{(5,3,3,3,2)} S$$

Example 3.5.2. *The charge of the tableau* $\begin{array}{ccc} 2&3& \\ 1&1&4 \end{array}$ *can be calculated by the following procedure:*

$$\begin{array}{ccc} 2&3& \\ 1&1&4 \end{array} \longrightarrow \begin{array}{ccccc} 2&3&&& \\ &&1&1&4 \end{array} \longrightarrow \begin{array}{ccccc} 4&&&& \\ &2&3&& \\ &&&1&1 \end{array} \longrightarrow \begin{array}{cc} 4& \\ 2&3 \end{array} \longrightarrow \begin{array}{cc} 3& \\ 1&2 \end{array}$$

$$c\left(\begin{array}{ccc} 2&3& \\ 1&1&4 \end{array}\right) = 1 + c\left(\begin{array}{cc} 3& \\ 1&2 \end{array}\right)$$

$$\begin{array}{cc} 3& \\ 1&2 \end{array} \longrightarrow \begin{array}{ccc} 3&& \\ &1&2 \end{array} \longrightarrow \begin{array}{ccc} 2&& \\ &3& \\ &&1 \end{array} \longrightarrow \begin{array}{cc} 2&3 \end{array} \longrightarrow \begin{array}{cc} 1&2 \end{array}$$

$$c\left(\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{2}\end{array}\right) = 1 + c\left(\boxed{1}\boxed{2}\right)$$

$$\boxed{1}\boxed{2} \quad\longrightarrow\quad \boxed{1}\boxed{2} \quad\longrightarrow\quad \begin{array}{ll}\boxed{2} & \\ & \boxed{1}\end{array} \quad\longrightarrow\quad \boxed{2} \quad\longrightarrow\quad \boxed{1}$$

$$c\left(\boxed{1}\boxed{2}\right) = 1 + c\left(\boxed{1}\right) = 1$$

Therefore the charge of the tableau $\begin{array}{l}\boxed{2}\boxed{3}\\\boxed{1}\boxed{1}\boxed{4}\end{array}$ *is* 3 *and* $\begin{array}{l}\boxed{2}\boxed{3}\\\boxed{1}\boxed{1}\boxed{4}\end{array} = \mathbf{H}_2^{(3,2)}\mathbf{H}_1^{(2,1)}\mathbf{H}_1^{(2)}\mathbf{H}_1^{(1)}\cdot$.

Example 3.5.3. *The tableau* $\begin{array}{l}\boxed{4}\\\boxed{2}\boxed{3}\boxed{5}\\\boxed{1}\boxed{1}\boxed{2}\boxed{6}\end{array}$ *is isomorphic to* $\mathbf{H}_2^{(10,3,1)}\begin{array}{l}\boxed{5}\\\boxed{3}\\\boxed{1}\boxed{1}\boxed{2}\boxed{4}\end{array}$ *(and is equal to* K_6 *applied to this tableau).*

$\begin{array}{l}\boxed{5}\\\boxed{3}\\\boxed{1}\boxed{1}\boxed{2}\boxed{4}\end{array}$ *is isomorphic to* $\mathbf{H}_2^{(8,1,1)}\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{2}\boxed{4}\end{array}$.

$\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{2}\boxed{4}\end{array}$ *is isomorphic to* $\mathbf{H}_1^{(5,1)}\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{2}\end{array}$.

$\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{2}\end{array}$ *is isomorphic to* $\mathbf{H}_1^{(4,1)}\boxed{1}\boxed{2}$.

$\boxed{1}\boxed{2}$ *is isomorphic to* $\mathbf{H}_1^{(3)}\boxed{1}$.

By abuse of notation $\begin{array}{l}\boxed{4}\\\boxed{2}\boxed{3}\boxed{5}\\\boxed{1}\boxed{1}\boxed{2}\boxed{6}\end{array} = \mathbf{H}_2^{(4,3,1)}\mathbf{H}_2^{(4,1,1)}\mathbf{H}_1^{(3,1)}\mathbf{H}_1^{(2,1)}\mathbf{H}_1^{(2)}\mathbf{H}_1^{(1)}\cdot$ *and the charge will be* $\sum_{i=1}^{6}\nu_1^{(i)} - 8 = 8$.

3.6 A list of tableau operations

Let $\lambda = \lambda(T)$ and $\mu = \mu(T)$. $\rho/\lambda \in \mathcal{H}_n$.

Table 3.1 gives a list of tableau operators mentioned in this paper and their effect on charge and cocharge. The charge and cocharge entries in the table of $C_\nu T$ and $C_\nu^{-1}T$ assume that the cell that is evacuated and inserted is not a 1.

Table 3.2 lists the tableau operators mentioned in this paper and gives the effect on the content and shape when that can be expressed in terms of the original

Operation	charge	cocharge
$(i, i+1)T$	$c(T) + \mu_i - \mu_{i+1}$	$co(T)$
$\theta_{\mu\nu}T$	$n(\nu) - n(\mu) + c(T)$	$co(T)$
$C_\nu T$	$c(T) + 1$	$co(T) - 1$
$C_\nu^{-1}T$	$c(T) - 1$	$co(T) + 1$
$K_n T$	$c(T)$	$co(T) - \frac{n(2\mu_1 - n - 1)}{2}$
$A_n^\rho T$	$c(T)$	$co(T) + \frac{n(2\mu_1 - n - 1)}{2}$
$\mathbf{H}_m^\rho T$	$c(T) + \lvert\lambda/\rho^r\rvert$	$co(T) + \lvert T\rvert - \lvert\lambda/\rho^r\rvert$
$R_n T$	$c(T) + \lvert T\rvert$	$co(T)$
$S_n T$	$c(T)$	$co(T) + \lvert T\rvert$

Table 3.1: A list of tableau operators and their effect on charge and cocharge

Operation	content	shape
$(i, i+1)T$	$(i, i+1)\mu$	λ
$\theta_{\mu\nu}T$	ν	λ
$C_\nu T$	μ	
$C_\nu^{-1}T$	μ	
$K_n T$	$(\mu_1 - n, \mu^r)$	
$A_n^\rho T$	$(\mu_1 + n, \mu^r)$	λ
$\mathbf{H}_m^\rho T$	$(m + \lvert T\rvert, \mu)$	ρ
$R_n T$	(n, μ)	$(n + \lambda_1, \lambda^r)$
$S_n T$	(n, μ)	(n, λ)

Table 3.2: A list of tableau operators and their effect on shape and content

tableau, T. The shape of $C_\nu(T)$, $C_\nu^{-1}(T)$, and $K_n(T)$ are not listed because the exact shape is dependent on more than the shape of T. It is known that $\lambda(C_\nu(T)) \leftarrow \nu$ and $\lambda(C_\nu^{-1}(T)) \leftarrow \nu$ and $\lambda(T)/\lambda(K_n(T)) \in \mathcal{H}_n$, but these entries in the table have been left blank.

If T is a standard tableau then the operation of reflecting the shape about the diagonal may be denoted by ωT. This is an involution that has the effect of interchanging the charge and the cocharge (that is, $c(\omega T) = co(T)$).

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