This worksheet is an implementation of the product rules that are in "*Products of characters of the symmetric group*" (arXiv:1709.08098 (https://arxiv.org/abs/1709.08098)) by Rosa Orellana and Mike Zabrocki

Abstract: In a recent paper (arXiv:1605.06672 (https://arxiv.org/abs/1605.06672)), the authors introduced a new basis of the ring of symmetric functions which evaluate to the irreducible characters of the symmetric group at roots of unity. The structure coefficients for this new basis are the stable Kronecker coefficients. In this paper we give combinatorial descriptions for several products that have as consequences several versions of the Pieri rule for this new basis of symmetric functions. In addition, we give several applications of the products studied in this paper.

In this worksheet we will compute a few examples of the three main theorems of the paper. These theorems provide a combinatorial rule for the product of characters of the symmetric group as symmetric functions in terms of certain types of multiset tableaux. The coefficients represent the multiplicity of the irreducibles a tensor product of symmetric group modules.

This worksheet is available in both
pdf (http://garsia.math.yorku.ca/~zabrocki/papers/stbasisPieriTableauxStructure.pdf) and
jupyter notebook (http://garsia.math.yorku.ca/~zabrocki/papers/stbasisPieriTableauxStructure.ipynb) form.
Note that the majority of the tableaux and multiset partition code that makes these functions run has not been integrated into Sage, but for the time being I am making these programs available on my website (subject to update and change):
http://garsia.math.yorku.ca/~zabrocki/msp.py (http://garsia.math.yorku.ca/~zabrocki/msp.py)
http://garsia.math.yorku.ca/~zabrocki/mst.py (http://garsia.math.yorku.ca/~zabrocki/mst.py)
Programs for calculating with the symmetric functions $s_{\tilde{\lambda}}$ and $h_{\tilde{\lambda}}$ is part of `SymmetricFunctions` code as a part of Sage.

In [1]:
```
load("http://garsia.math.yorku.ca/~zabrocki/msp.py")
load("http://garsia.math.yorku.ca/~zabrocki/mst.py")
```

```
Loading MultiSetPartition, SetPartitionofMultiSet, MultiSetPartition
s and SetPartitionsofMultiSet
BiSetPartition, BiSetPartitions, MCPartition, MCPartitions (v 5.0 da
te: Jan 7, 2018)
grlex_msp, lex_msp, gr_rev_lex_msp, rev_lex_msp, rev_gr_revlex_msp
Loading MultiSetTableau, MultiSetTableaux, SetTableau, SetTableaux,
BiSetTableau and BiSetTableaux, MCTableau, MCTableaux (v 4.0 date: J
an 7, 2018)
```

In [2]:
```
SymmetricFunctions(QQ).inject_shorthands('all')
```

```
Defining e as shorthand for Symmetric Functions over Rational Field
in the elementary basis
Defining f as shorthand for Symmetric Functions over Rational Field
in the forgotten basis
Defining h as shorthand for Symmetric Functions over Rational Field
in the homogeneous basis
Defining ht as shorthand for Symmetric Functions over Rational Field
in the induced trivial character basis
Defining m as shorthand for Symmetric Functions over Rational Field
in the monomial basis
Defining o as shorthand for Symmetric Functions over Rational Field
in the orthogonal basis
Defining p as shorthand for Symmetric Functions over Rational Field
in the powersum basis
Defining s as shorthand for Symmetric Functions over Rational Field
in the Schur basis
Defining sp as shorthand for Symmetric Functions over Rational Field
in the symplectic basis
Defining st as shorthand for Symmetric Functions over Rational Field
in the irreducible symmetric group character basis
Defining w as shorthand for Symmetric Functions over Rational Field
in the Witt basis
```

In [ ]:

Definition 1. Let γ be a partition, α and β be compositions. Then the set $\mathcal{MCT}_\gamma(\alpha,\beta)$ contains tableaux T that are column strict with respect to the reverse lex order, have shape $(r,\gamma)/(\gamma_1)$, and content $\{\!\{\overline{1}^{\alpha_1},\overline{2}^{\alpha_2},\ldots,\overline{\ell}^{\alpha_\ell},1^{\beta_1},2^{\beta_2},\ldots,k^{\beta_k}\}\!\}$ with at most one barred entry in each cell. Further, we require that cells in the first row cannot be filled with multisets containing only barred entries.

In [3]:
```
MCTableaux([3,2],[3,3],[2,1,1],rlex_bsp)
```

Out[3]:
```
multiset common tableaux with shape [x]+[3, 2] and content vectors [
3, 3] and [2, 1, 1]
```

In [4]:
```
#optional: At least some of the tableaux options are followed by this
class (including 'convention')
Tableaux.options(convention='French')
```

In [5]:
```
MCTableaux([3,2],[3,3],[2,1],rlex_bsp).cardinality()
```

Out[5]:
```
2453
```

In [6]:
```
MCTableaux([3,2],[3,3],[2,1],rlex_bsp).first().pp()
```

```
2  2
-1 -1 -2
.  .  .  1 1 1 2
```

In [7]:
```
# Here is an example of a set of tableaux that can be computed for thi
s class
for T in MCTableaux([1,1],[2],[1,1],rlex_bsp):
    T.pp()
```

```
-2
-1
.  1 1

-2
-1
.  11

-11
-2
.   1

1
-2
.  -11

-111
-2
.

-21
-1
.   1

1
-1
.  -21

-211
-1
.

-21
-11
.
```

In [ ]:

In [ ]:

Definition 3. If $T \in \mathcal{MCT}_\gamma(\lambda, \mu)$ for some partitions λ, μ and γ, then let $S_1 < S_2 < \ldots < S_d$ be the multisets of the unbarred entries that appear in T (ignoring the barred entries), then we say that T is a *lattice tableau* if the word

$$read(T|_-)read(T|_{S_1})read(T|_{S_2})\cdots read(T|_{S_d})$$

is lattice.

In [8]:
```
Tlist = MCTableaux([3,2],[2],[2,2],rlex_bsp).list()
```

In [9]:
```
len(Tlist)
```

Out[9]:
```
10
```

In [10]:
```
# some examples of the negative reading word is NOT lattice
for T in Tlist:
    if not T.neg_reading_word_is_lattice():
        T.pp()
        print "reading word = ", T.neg_reading_word()
```

```
-2 1
-1 -1 -2
.  .  .  1

reading word =  [-2, -1, -1, -2]
-2 11
-1 -1 -2
.  .  .

reading word =  [-2, -1, -1, -2]
1  -11
-1 -2  -2
.  .   .

reading word =  [-2, -2, -1, -1]
-2 -11
-1 -2  1
.  .   .

reading word =  [-2, -1, -2, -1]
-2 1
-1 -2 -11
.  .  .

reading word =  [-2, -1, -2, -1]
1  -21
-1 -1  -2
.  .   .

reading word =  [-2, -1, -1, -2]
```

In [11]:
```
for T in Tlist:
    if T.neg_reading_word_is_lattice():
        T.pp()
        print "reading word = ", T.neg_reading_word()
```

```
-2 -2
-1 -1 1
 .  .  . 1

reading word =  [-1, -1, -2, -2]
-2 -2
-1 -1 11
 .  .  .

reading word =  [-1, -1, -2, -2]
-2 -21
-1 -1  1
 .  .   .

reading word =  [-1, -1, -2, -2]
-2 1
-1 -1 -21
 .  .  .

reading word =  [-1, -1, -2, -2]
```

In [ ]:

In [ ]:

Theorem 13. *Let λ and γ be partitions and α a composition, then the coefficient of $\tilde{s}_\gamma$ in $h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}} \tilde{s}_\lambda$ is equal to the number of $T \in \mathcal{MCT}_\gamma(\lambda, \alpha)$ such that T is a lattice tableau.*

In [12]:
```
# here is a single coefficient in the product from Theorem 13
(st[2,1]*h[3,2,1]).coefficient([3,2])
```

Out[12]: 3776

In [13]:
```
# and here it is computed combinatorially
len([T for T in MCTableaux([3,2],[3,2,1],[2,1],rlex_bsp) if T.neg_read
ing_word_is_lattice()])
```

Out[13]: 3776

In [14]:
```
# Consider the following symmetric function expression
st[1,1]*h[2,1]
```

Out[14]:
```
3*st[] + 15*st[1] + 25*st[1, 1] + 17*st[1, 1, 1] + 4*st[1, 1, 1, 1]
+ 20*st[2] + 27*st[2, 1] + 11*st[2, 1, 1] + st[2, 1, 1, 1] + 6*st[2,
2] + st[2, 2, 1] + 10*st[3] + 9*st[3, 1] + 2*st[3, 1, 1] + st[3, 2]
+ 2*st[4] + st[4, 1]
```

In [15]:
```
# and this may be computed combinatorially with the following sum
sum(len([T for T in MCTableaux(la,[2,1],[1,1],rlex_bsp) if T.neg_readi
ng_word_is_lattice()])*st(la) \
    for d in range(6) for la in Partitions(d))
```

Out[15]:
```
3*st[] + 15*st[1] + 25*st[1, 1] + 17*st[1, 1, 1] + 4*st[1, 1, 1, 1]
+ 20*st[2] + 27*st[2, 1] + 11*st[2, 1, 1] + st[2, 1, 1, 1] + 6*st[2,
2] + st[2, 2, 1] + 10*st[3] + 9*st[3, 1] + 2*st[3, 1, 1] + st[3, 2]
+ 2*st[4] + st[4, 1]
```

In [ ]:

In [ ]:

Theorem 14. *Let λ and γ be partitions and α a composition, then the coefficient of $\tilde{s}_\gamma$ in $\tilde{h}_{\alpha_1} \tilde{h}_{\alpha_2} \cdots \tilde{h}_{\alpha_{\ell(\alpha)}} \tilde{s}_\lambda$ is equal to the number of $T \in \mathcal{MCT}_\gamma(\lambda, \alpha)$ such that the entries of the tableaux are sets (no repeated entries) and T is a lattice tableau.*

In [16]:
```
# Here is a demonstration of the calculation of a single coefficient f
rom Theorem 14
(st[2,1]*ht[3]*ht[2]*ht[1]).coefficient([3,2])
```

Out[16]: 1730

In [17]:
```
# and here it is computed combinatorially
len([T for T in MCTableaux([3,2],[3,2,1],[2,1],rlex_bsp) if T.neg_read
ing_word_is_lattice() and T.has_set_entries()])
```

Out[17]:
```
1730
```

In [18]:
```
st[1,1]*ht[2]*ht[1]
```

Out[18]:
```
2*st[] + 10*st[1] + 17*st[1, 1] + 12*st[1, 1, 1] + 3*st[1, 1, 1, 1]
+ 14*st[2] + 20*st[2, 1] + 9*st[2, 1, 1] + st[2, 1, 1, 1] + 5*st[2,
2] + st[2, 2, 1] + 8*st[3] + 8*st[3, 1] + 2*st[3, 1, 1] + st[3, 2] +
2*st[4] + st[4, 1]
```

In [19]:
```
# and this may be computed combinatorially with the following sum
sum(len([T for T in MCTableaux(la,[2,1],[1,1],rlex_bsp) if T.neg_readi
ng_word_is_lattice()\
         and T.has_set_entries()])*st(la) for d in range(6) for la in
Partitions(d))
```

Out[19]:
```
2*st[] + 10*st[1] + 17*st[1, 1] + 12*st[1, 1, 1] + 3*st[1, 1, 1, 1]
+ 14*st[2] + 20*st[2, 1] + 9*st[2, 1, 1] + st[2, 1, 1, 1] + 5*st[2,
2] + st[2, 2, 1] + 8*st[3] + 8*st[3, 1] + 2*st[3, 1, 1] + st[3, 2] +
2*st[4] + st[4, 1]
```

In [ ]:

In [ ]:

Theorem 15. *Let λ and γ be partitions and α a composition, then the coefficient of $\tilde{s}_\gamma$ in $\tilde{s}_{\alpha_1}\tilde{s}_{\alpha_2}\cdots\tilde{s}_{\alpha_{\ell(\alpha)}}\tilde{s}_\lambda$ is equal to the number of $T \in \mathcal{MCT}_\gamma(\lambda,\alpha)$ such that the entries of the tableaux are sets (no repeated entries), T is a lattice tableau, and only labels of sets of size greater than 1 are allowed in the first row.*

In [20]:
```
# Here is a demonstration of the calculation of a single coefficient f
rom Theorem 15
(st[2,1]*st[3]*st[2]*st[1]).coefficient([3,2])
```

Out[20]:
```
757
```

In [21]:
```
# and here it is computed combinatorially
len([T for T in MCTableaux([3,2],[3,2,1],[2,1],rlex_bsp) if T.neg_read
ing_word_is_lattice()\
      and T.has_set_entries() and T.has_first_row_entries_size_greater_
than_one()])
```

Out[21]:
```
757
```

In [22]:
```
st[1,1]*st[2]*st[1]
```

Out[22]:
```
st[] + 4*st[1] + 7*st[1, 1] + 6*st[1, 1, 1] + 2*st[1, 1, 1, 1] + 7*s
t[2] + 11*st[2, 1] + 6*st[2, 1, 1] + st[2, 1, 1, 1] + 4*st[2, 2] + s
t[2, 2, 1] + 5*st[3] + 6*st[3, 1] + 2*st[3, 1, 1] + st[3, 2] + 2*st[
4] + st[4, 1]
```

In [23]:
```
# and this may be computed combinatorially with the following sum
sum(len([T for T in MCTableaux(la,[2,1],[1,1],rlex_bsp) if T.neg_readi
ng_word_is_lattice()\
         and T.has_set_entries() and T.has_first_row_entries_size_grea
ter_than_one()])*st(la) \
        for d in range(6) for la in Partitions(d))
```

Out[23]:
```
st[] + 4*st[1] + 7*st[1, 1] + 6*st[1, 1, 1] + 2*st[1, 1, 1, 1] + 7*s
t[2] + 11*st[2, 1] + 6*st[2, 1, 1] + st[2, 1, 1, 1] + 4*st[2, 2] + s
t[2, 2, 1] + 5*st[3] + 6*st[3, 1] + 2*st[3, 1, 1] + st[3, 2] + 2*st[
4] + st[4, 1]
```

In [ ]:

In [ ]: