Plethysm, symmetric chain decompositions and Ehrhart polynomials

Mike Zabrocki
Joint work with Álvaro Gutiérrez, Rosa Orellana, Franco Saliola, Anne Schilling

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Outline

1. **What is plethysm?**
2. **plethysm → symmetric chains**
3. **machine learning symmetric chains**
4. **Ehrhart theory and generating functions of plethysm coefficients**

What is .... plethysm?

a composition of Schur functors

Schur functor $$\mathbb{S}^{\lambda}(V) := \mathrm{Hom}_{S_d}(W^{\lambda}_{S_d}, V^{\otimes d})$$

note: result is 0 if $\ell(\lambda) > \dim(V)$

think of $\mathbb{S}^{\lambda}(V)$ as "column strict tableaux shape λ whose entries are basis of V"

plethysm

$$\boxed{\mathbb{S}^{\lambda}(\mathbb{S}^{\mu}(V)) := \bigoplus_{\substack{\gamma \vdash |\lambda||\mu| \\ \ell(\gamma) \leq \dim(V)}} \mathbb{S}^{\gamma}(V)^{\oplus a^{\gamma}_{\lambda[\mu]}}}$$

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plethysm coefficient

How does one compute plethysm?

(1936)

characters
symmetric functions
transition coefficients

In algebra, **plethysm** is an operation on symmetric functions introduced by Dudley E. Littlewood,[1] who denoted it by $\{\lambda\} \otimes \{\mu\}$. The word "plethysm" for this operation (after the Greek word πληθυσμός meaning "multiplication") was introduced later by Littlewood (1950, p. 289, 1950b, p.274), who said that the name was suggested by M. L. Clark.

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power sum definition

$$p_\lambda[cp_\mu] = c^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\mu)} p_{\lambda_i \mu_j}$$

$$s_\lambda = \sum_\nu \frac{\chi^\lambda(\nu)}{z_\nu} p_\nu \qquad p_\nu = \sum_\lambda \chi^\lambda(\nu) s_\lambda$$

$$s_\lambda[s_\mu] = \sum_{\gamma \vdash |\lambda||\mu|} a^\gamma_{\lambda[\mu]} s_\gamma$$

Computation of plethysm in Sage:

```
[sage: s = SymmetricFunctions(QQ).schur()
[sage: s[3](s[2,1])
 s[3, 2, 2, 1, 1] + s[3, 2, 2, 2] + s[3, 3, 1, 1, 1] + s[3, 3, 2, 1] + s[3, 3, 3] + s[4, 1, 1, 1, 1, 1] +
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$$a^{441}_{3[21]} = 1$$

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$$a_{3[21]}^{441} = 1$$

Goal: give some combinatorial description of these non-negative integers, or, failing that, develop faster methods for computing these coefficients.

The monomial and fundamental expansion of plethysm:

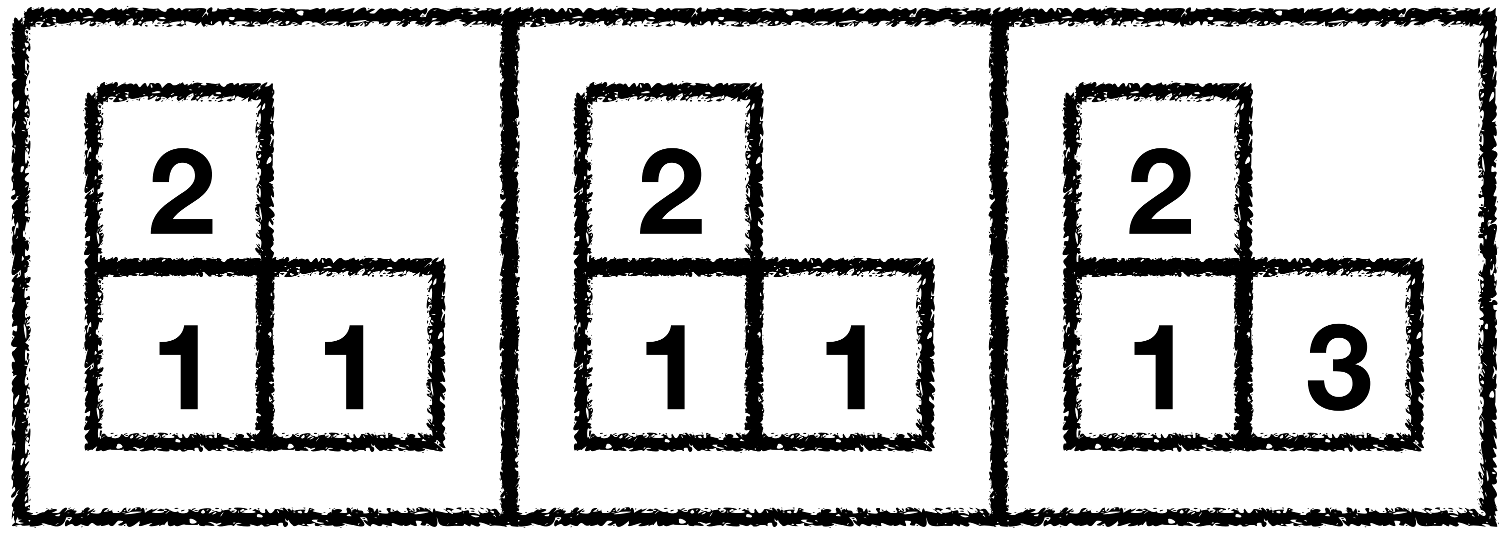
$$s_\lambda[s_\mu](x_1, x_2, \ldots, x_n) = \sum_T \mathbf{x}^{wt(T)}$$

T are column strict tableaux of shape λ
with entries that are column strict tableaux of shape μ

French convention on tableaux

Example:

$$s_3[s_{21}]$$



$$wt(T) = x_1^6 x_2^3 x_3^1$$

general open problem: how does one (combinatorially) pass from the monomial
or fundamental expansion of a symmetric function expression to the Schur expansion?

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Fundamental expansion due to Loehr-Warrington (2010)

- one fundamental term for each standard tableau of tableaux

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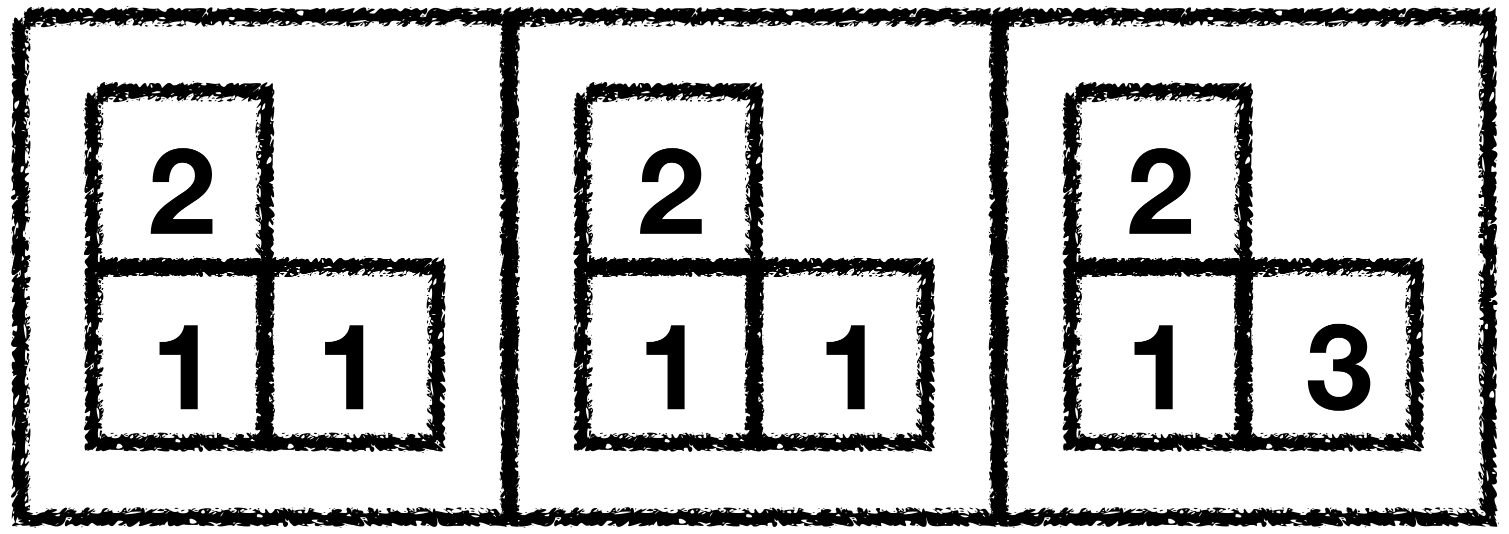

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+ sign reversing involution and identify the fixed points

Next method: crystals

first non-trivial case:

solve the problem for two variables

$a^{(\nu_1,\nu_2)}_{w[h]}$

focus on partitions $\lambda = (w)$ and $\mu = (h)$ for $w, h \in \mathbb{N}$

Monomial expansion in the case of $s_3[s_4](1, q)$

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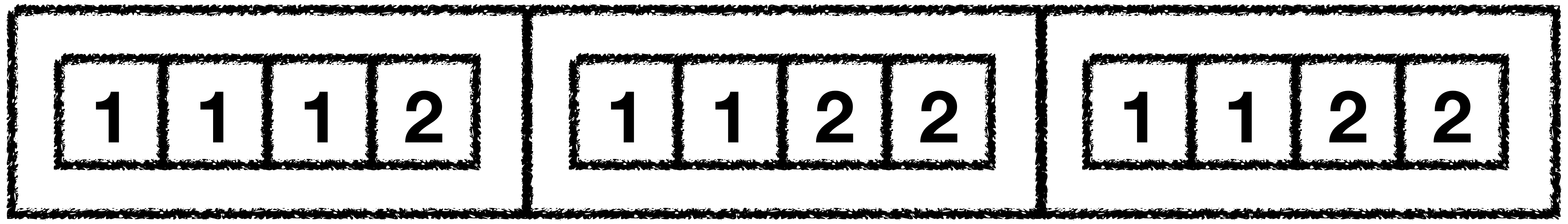

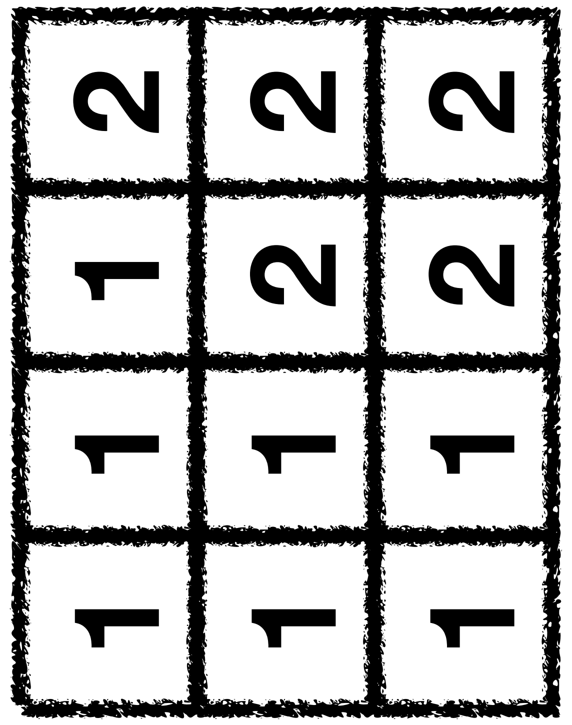

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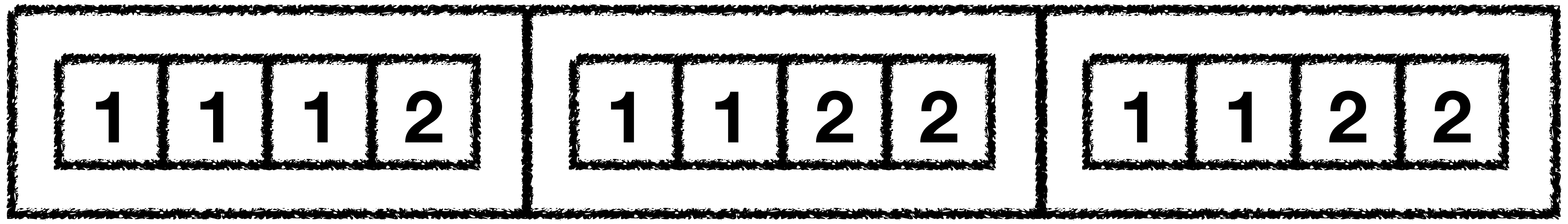

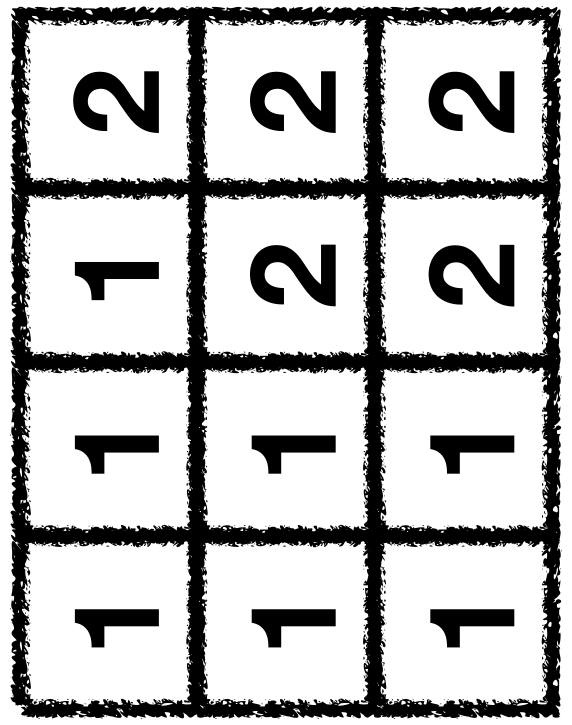

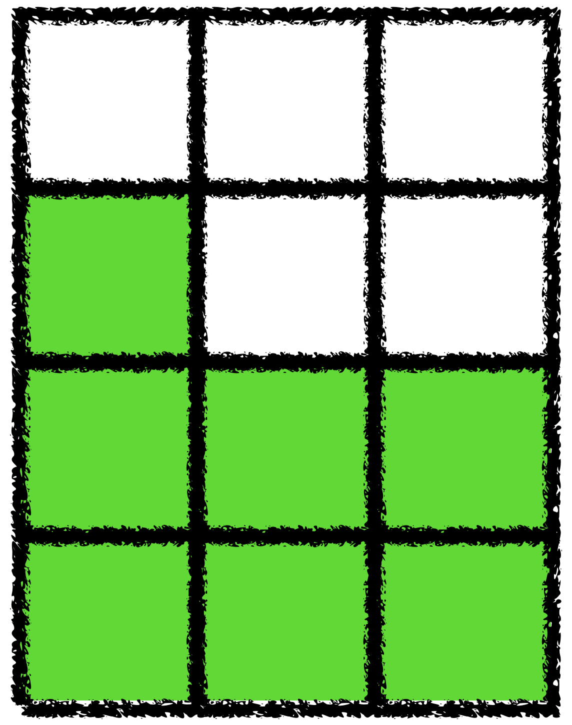

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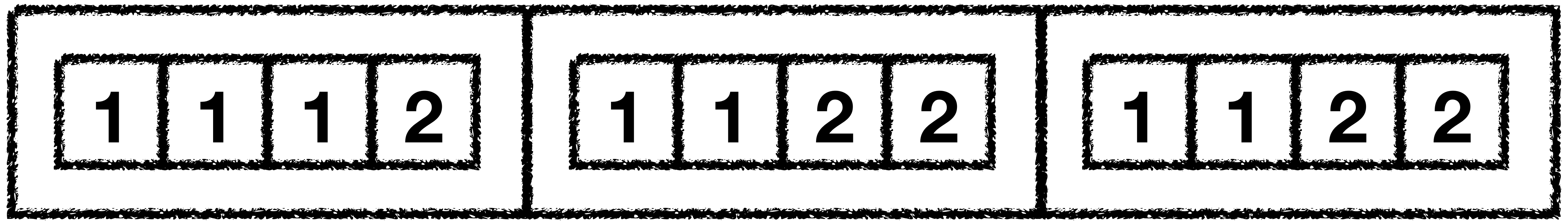
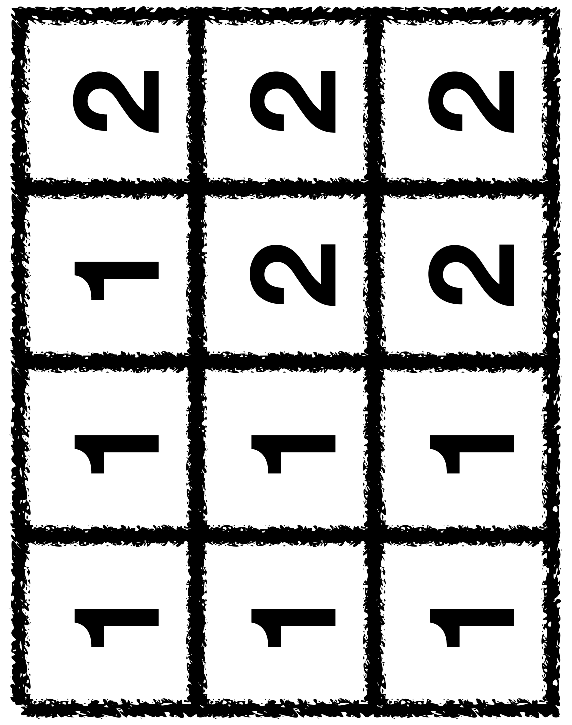
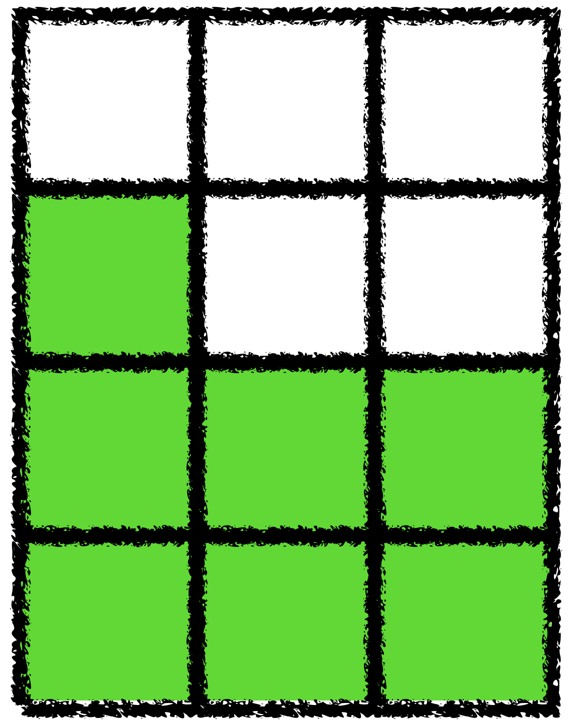


$$s_w[s_h](1,q) = \sum_{\gamma \subseteq (w^h)} q^{|\gamma|} = \begin{bmatrix} w+h \\ h \end{bmatrix}_q$$





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$$s_\mu(1,q) = \begin{cases} 0 & \text{if } \ell(\mu) > 2 \\ q^{\mu_2} + q^{\mu_2+1} + \cdots + q^{\mu_1} & \text{if } \ell(\mu) \leq 2 \end{cases}$$

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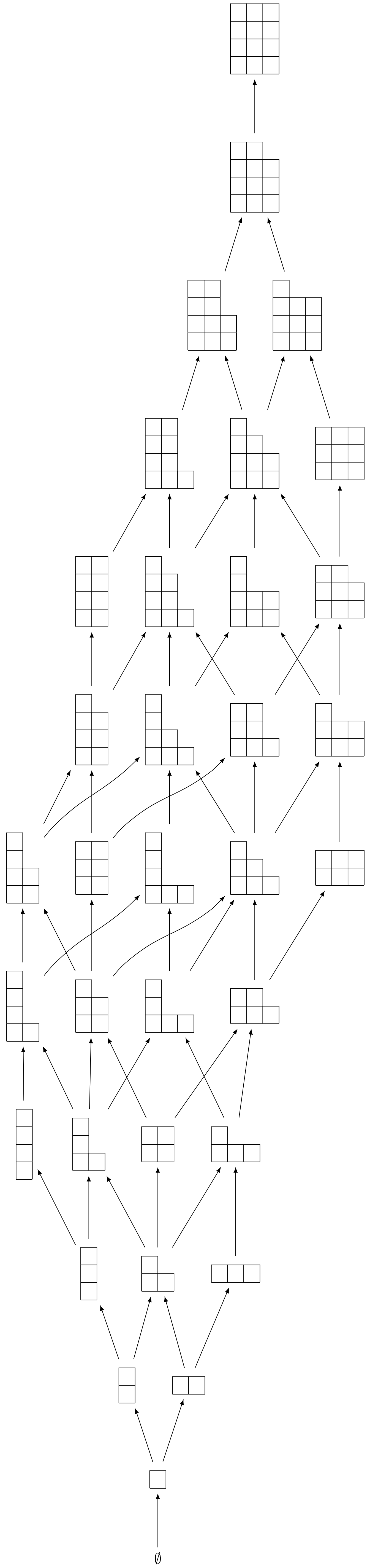
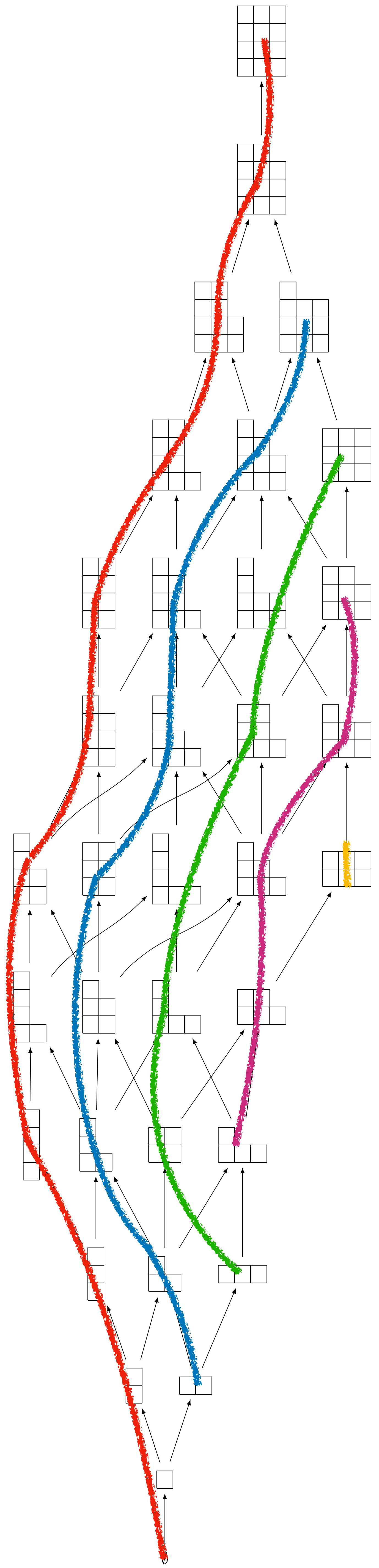


$$s_3[s_4](1,q) = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12}$$



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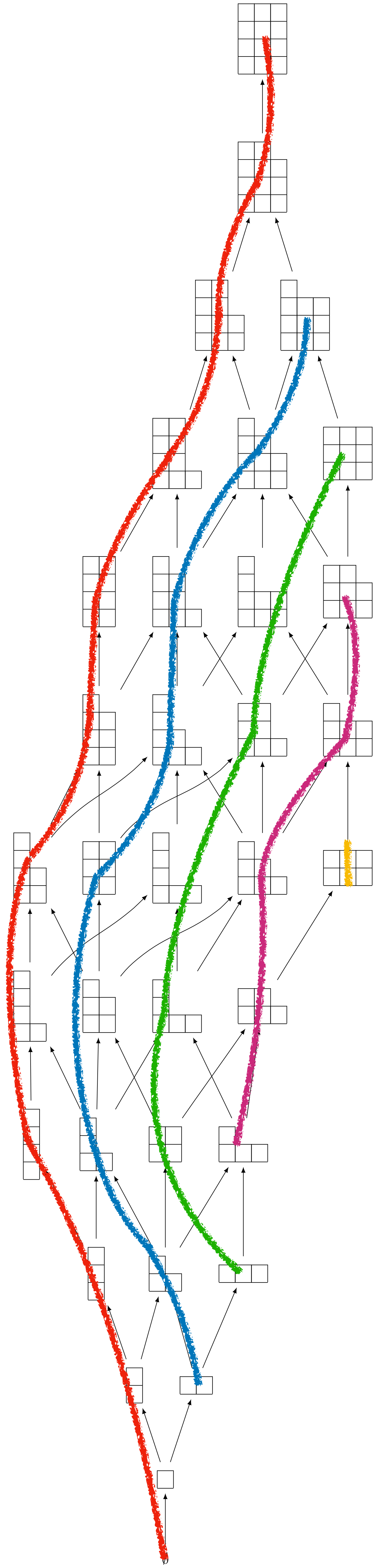

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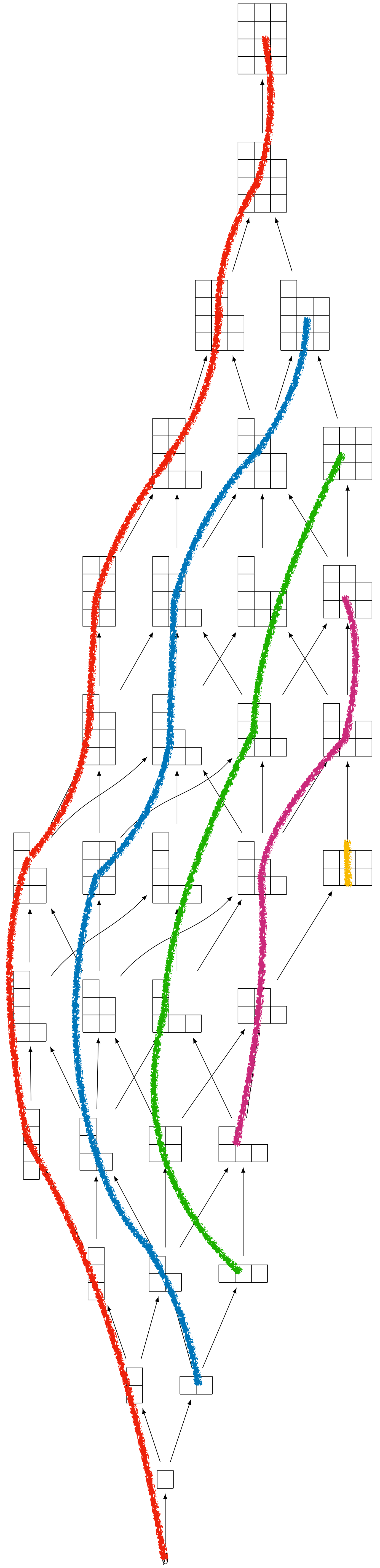

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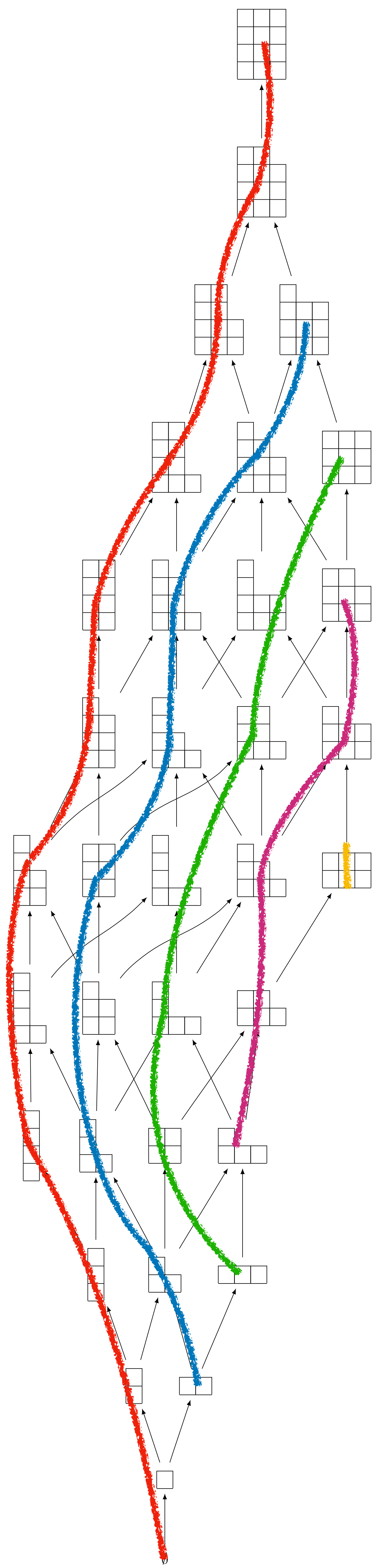
Goal: Combinatorially describe the highest or lowest weights of these chains

Ideally: clearly describe the whole graph structure of one set of chains that follows the poset structure of the set of partitions (symmetric chain decomposition of the poset).



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L(*w*,*h*) = poset of partitions in a *w x h* rectangle

symmetric chain decompositions are known for w=1,2,3,4 (and 5)
(see Álvaro’s talk from FPSAC ’25)

This is a weird problem because for any fixed *w* and *h*, large numbers of symmetric chain decompositions (SCD) seem to exist. Proving that one (human describable) SCD always exists seems to be hard.

Select a single SCD from many:

Side story, ask an LLM how to do this:

Let $L(w,h)$ be the set of integer partitions in a rectangle of width w and height h. For $d < wh/2$, what is an injective embedding of the partitions of size d in $L(w,h)$ into the partitions of size $d+1$?

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Thought for 1m 34s

Here's a clean, explicit injection that works uniformly for all w, h and all $d < \frac{wh}{2}$.

Think of a partition $\lambda \in L(w, h)$ as the monotone lattice path from $(0, 0)$ to (w, h) that traces the southeast boundary of its Ferrers diagram, encoded as a word $u(\lambda)$ in the letters E (step right) and N (step up) with exactly w E's and h N's. The size $|\lambda|$ equals the number of unit squares under that path, which is the number of ordered pairs (E, N) with the E occurring before the N in the word.

The map Φ

Given λ with $|\lambda| = d$, define $\Phi(\lambda)$ by finding the **leftmost** occurrence of the pattern NE in $u(\lambda)$ and swapping it to EN. Equivalently, on the Ferrers diagram boundary, this is the **southwest-most inner corner** (a vertical step followed by a horizontal step); replace that corner by a horizontal step followed by a vertical step. This adds exactly one box to the diagram, so $|\Phi(\lambda)| = d + 1$.

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Let w=3 and h=4. the partition (1,1,1,1) corresponds to the word ENNNNEE. What partition would that be mapped to in this embedding?

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Doesn't that make this map not injective because we have two words ENNNNEE and NENNENE mapping to the same word ENNNENE?

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Think of a partition $\lambda \in L(w,h)$ as the monotone lattice path from $(0,0)$ to (w,h) that traces the southeast boundary of its Ferrers diagram, encoded as a word $u(\lambda)$ in the letters E (step right) and N (step up) with exactly w E's and h N's. The size $|\lambda|$ equals the number of unit squares under that path, which is the number of ordered pairs (E, N) with the E occurring before the N in the word.

The map Φ

Given λ with $|\lambda| = d$, define $\Phi(\lambda)$ by finding the **leftmost** occurrence of the pattern NE in $u(\lambda)$ and swapping it to EN. Equivalently, on the Ferrers diagram boundary, this is the **southwest-most inner corner** (a vertical step followed by a horizontal step); replace that corner by a horizontal step followed by a vertical step. This adds exactly one box to the diagram, so $|\Phi(\lambda)| = d + 1$.

Let w=3 and h=4. the partition (1,1,1,1) corresponds to the word ENNNNEE. What partition would that be mapped to in this embedding?

$\boxed{(2,1,1,1)}$.

Again for w=3 and h=4 and the partition (2,1,1) we would have the word NENNENE. What partition would that be mapped to in this embedding?

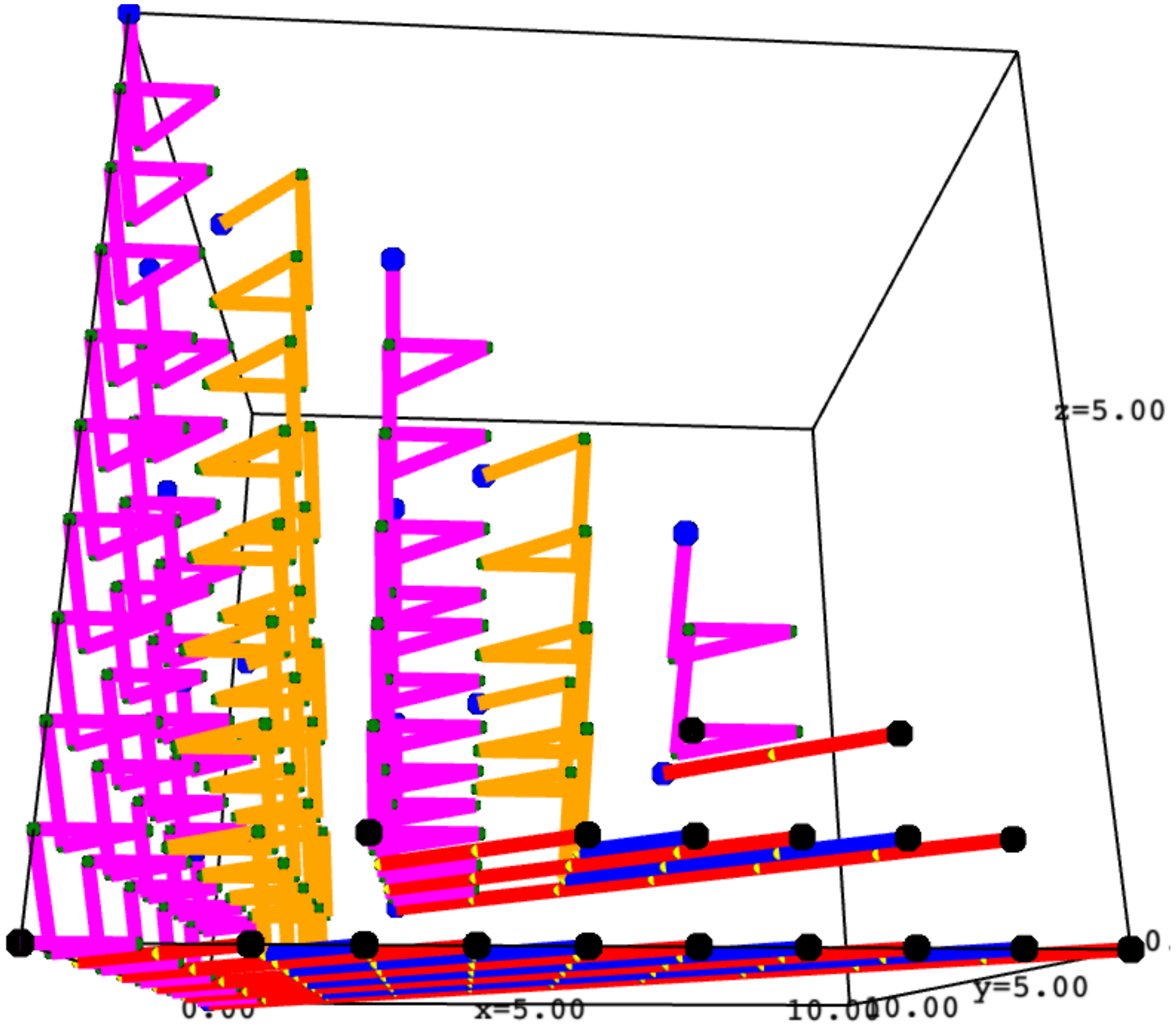
$\boxed{(2,1,1,1)}$.

Doesn't that make this map not injective because we have two words ENNNNEE and NENNENE mapping to the same word ENNNENE?

Thought for 40s

You're right — good catch. I made a mistake.

An answer for small *w*:



w = *3* and *h* = *10*

Geometry of points $(h \geq p_1 \geq p_2 \geq \cdots \geq p_w)$

Assume that chains with fixed width *w* and height *h* extend to the chains of width w and max height *h+1*

+ some additional minor assumptions characterizes the chains for *w* = *3* and *w* = *4*

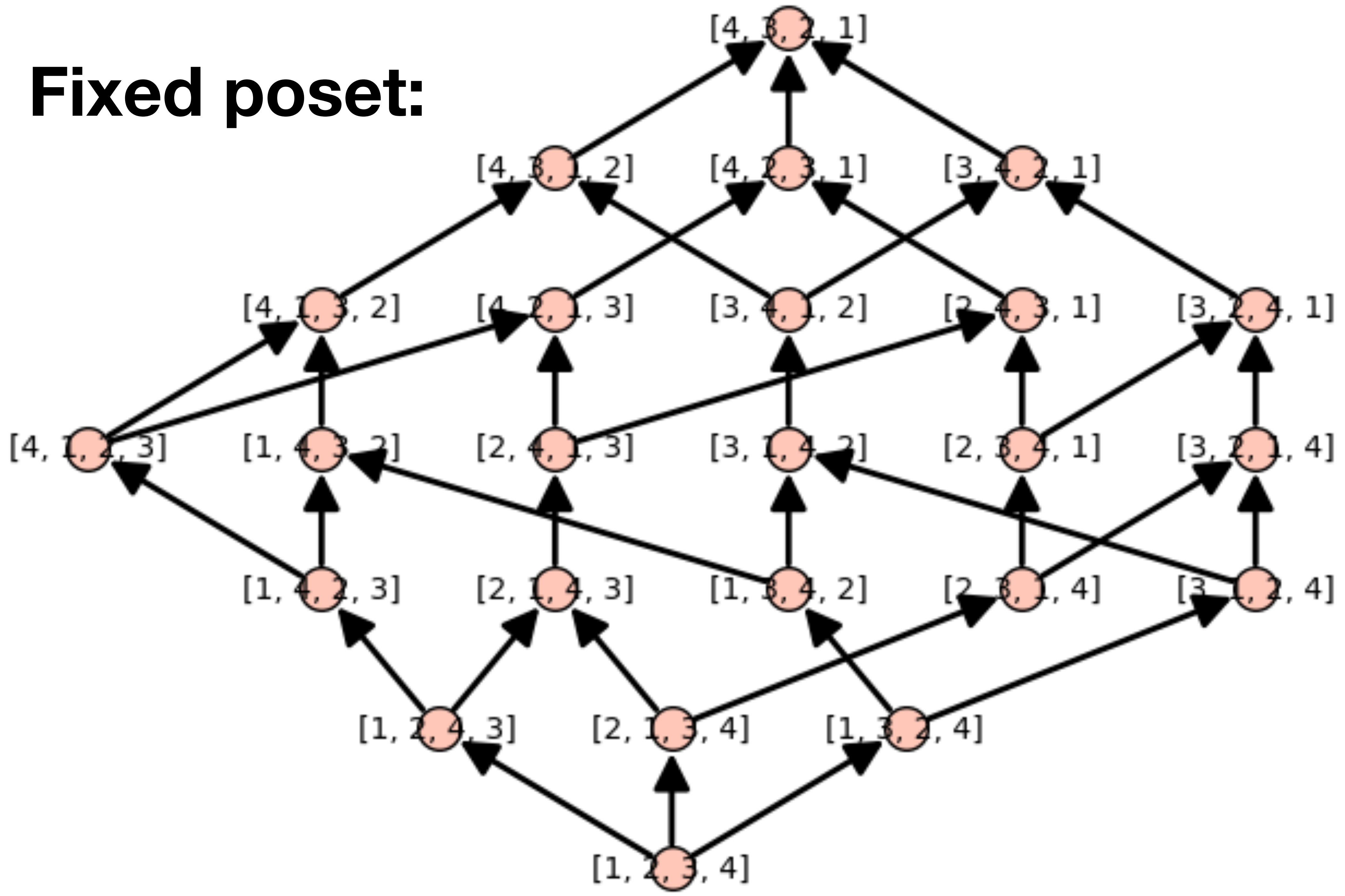
at w = 5 the conditions that we impose as humans to understand one family of chains fails to characterize and the result creates **too much data**

Data science approach to SCDs:

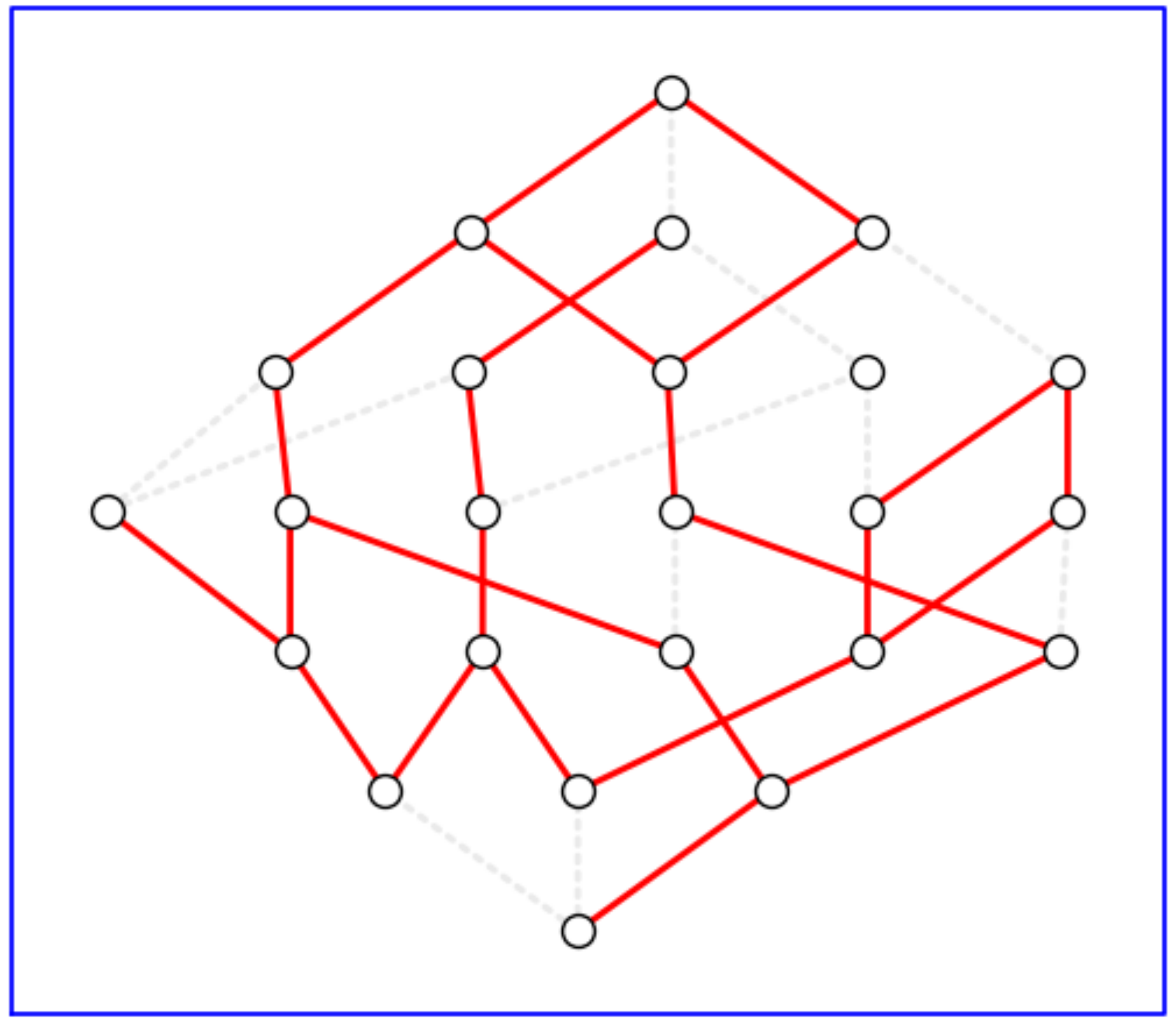
Use “reinforcement learning” algorithm to teach a neural network what a “good” SCD looks like

Example how to teach a neural network what an SCD looks like:

Fixed poset:



start selection of edges:



(0,1)-vector representing a selection of cover relation

Create a score function:

Idea: Don’t tell the neural network what an SCD is/is not,
instead tell it what you’d like to see/not like to see in a symmetric chain

Good:

Bad:

Create a score function:

Idea: Don’t tell the neural network what an SCD is/is not,
instead tell it what you’d like to see/not like to see in a symmetric chain

Good:

configurations with at most one incoming/outgoing edge 

Bad:

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Good:

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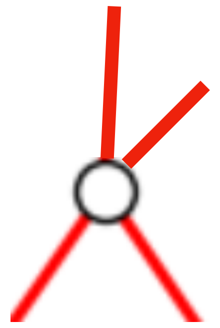
Bad:

large number of incoming/outgoing edges



Create a score function:

Idea: Don't tell the neural network what an SCD is/is not,
instead tell it what you'd like to see/not like to see in a symmetric chain

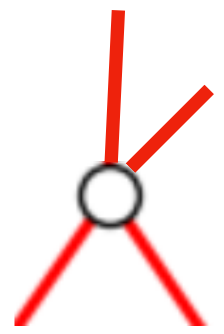
Good:

configurations with at most one incoming/outgoing edge

max rank - min rank in connected component is big

Bad:

large number of incoming/outgoing edges



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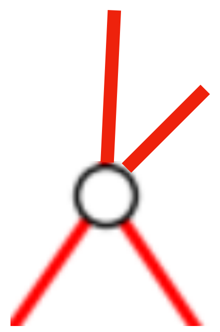
large number of incoming/outgoing edges 

configurations with one incoming and multiple outgoing 

Create a score function:

Idea: Don't tell the neural network what an SCD is/is not,
instead tell it what you'd like to see/not like to see in a symmetric chain

Good:

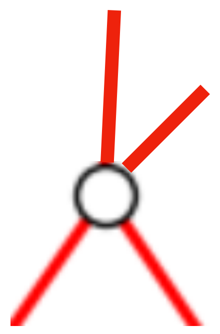
configurations with at most one incoming/outgoing edge

max rank - min rank in connected component is big

number of connected components = expected number of symmetric chains

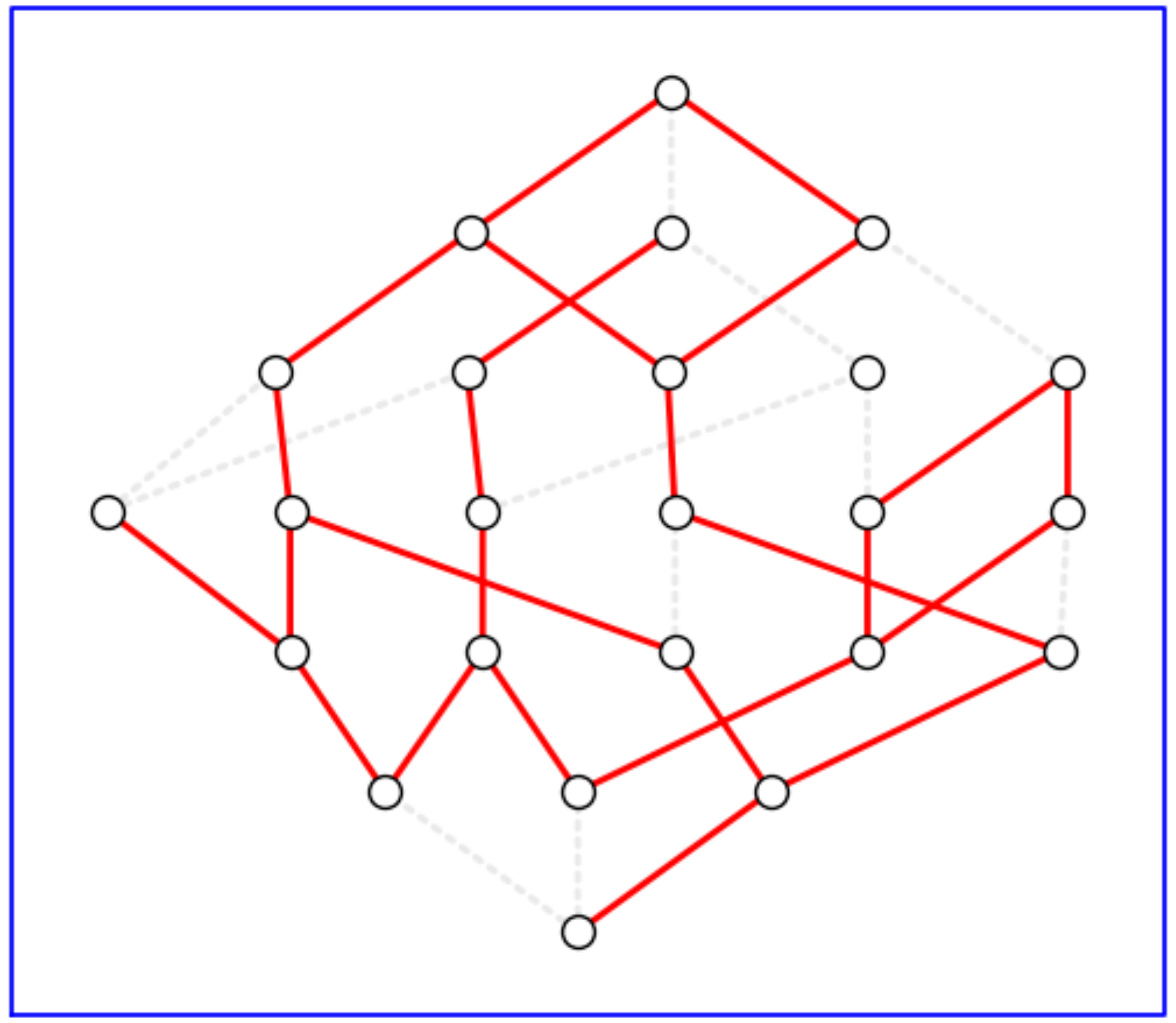
Bad:

large number of incoming/outgoing edges



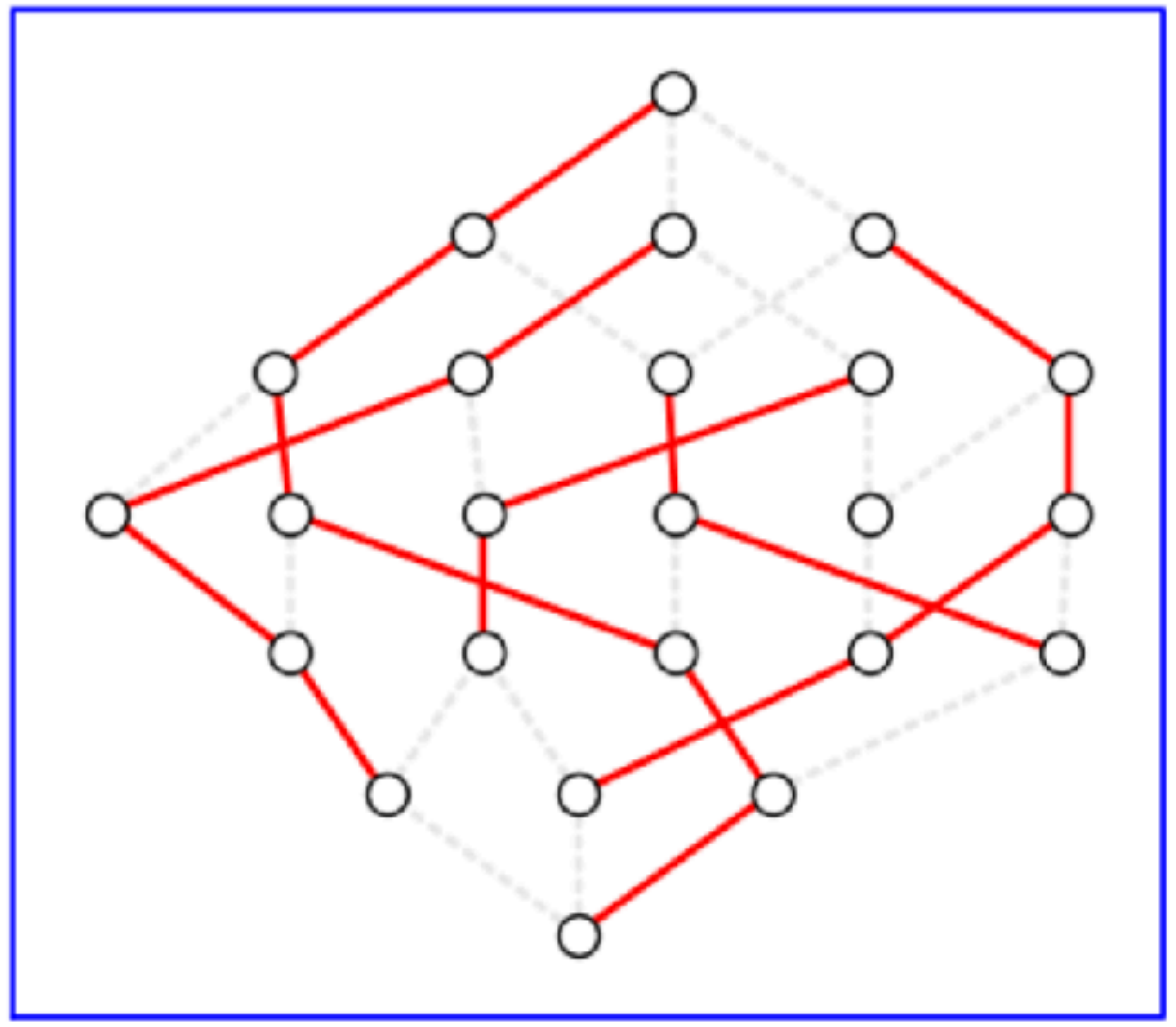
configurations with one incoming and multiple outgoing

start selection of edges:



train NN with
score function and
select candidates
that give a better score

ending selection of edges:



Remarks:

- We might be able to use this machine learning technique to extract a single human comprehensible SCD is for a fixed w and h

- Challenge #1 is that we don't have a clear notion of a "good" SCD of $L(w, h)$ what should we weight positively or negatively?

- Challenge #2 would be proving that the SCD can be extended to all h

- To solve the original goal of giving a combinatorial description of the plethysm coefficients, we don't need an SCD of $L(w,h)$, we just need a description of some analogue of the highest weights.

Connection to Ehrhart theory:

$$\mathcal{P} = \{(p_1, p_2, \ldots, p_w) \in \mathbb{R}^w : 1 \geq p_1 \geq p_2 \geq \cdots \geq p_w \geq 0\}$$

$$\mathrm{ehr}_{\mathcal{P}}(q,h) = \sum_{\mathbf{m} \in h\mathcal{P} \cap \mathbb{Z}^w} q^{|\mathbf{m}|} = \begin{bmatrix} w+h \\ h \end{bmatrix} = s_w[s_h](1,q)$$

$$\mathrm{Ehr}_{\mathcal{P}}(z;q) = \sum_{h \geq 0} \mathrm{ehr}(q,h) z^h = \sum_{h \geq 0} s_w[s_h](1,q) z^h$$

$$= \frac{1}{(1-z)(1-qz)(1-q^2z)\cdots(1-q^wz)}$$

Connection to Ehrhart theory:

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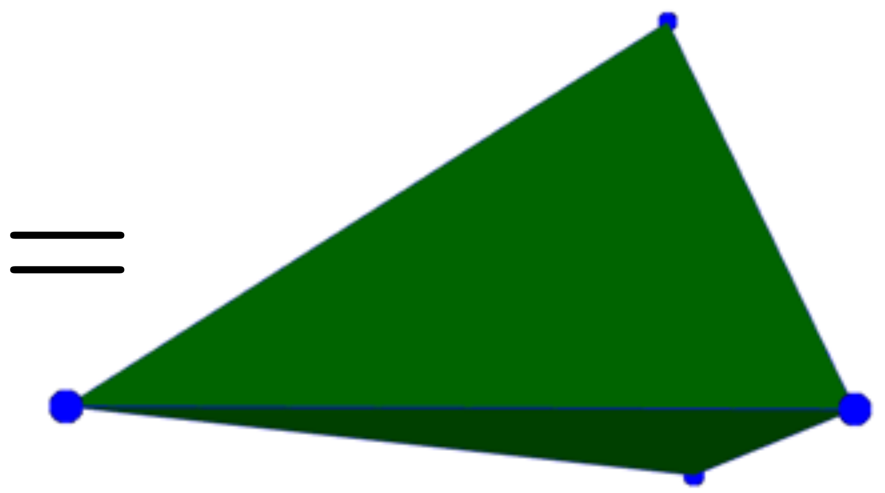
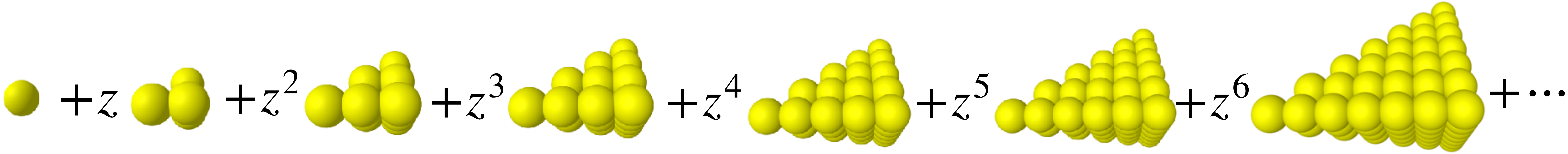
$$= \frac{1}{(1-z)(1-qz)(1-q^2z)\cdots(1-q^wz)}$$

Example w=3:

$\mathcal{P} =$ 



$$Ehr_{\mathcal{P}}(z;q) = \sum_{h\geq 0} s_w[s_h](1,q)z^h$$

$$(1-q)s_w[s_h](1,q) = a_{w[h]}^{(wh)} + qa_{w[h]}^{(wh-1,1)} + q^2 a_{w[h]}^{(wh-2,2)} + \cdots - q^{wh-1}a_{w[h]}^{(wh-2,2)} - q^{wh}a_{w[h]}^{(wh-1,1)} - q^{wh+1}a_{w[h]}^{(wh)}$$

Difference of Ehrhart series:

$$(1-q)Ehr_{\mathcal{P}}(z;q) = \sum_{h\geq 0} z^h \left(\sum_{k=0}^{\lfloor wh/2 \rfloor} a_{w[h]}^{(wh-k,k)} q^k - \sum_{k=0}^{\lfloor wh/2 \rfloor} a_{w[h]}^{(wh-k,k)} q^{wh+1-k} \right)$$

+ application of MacMahon partition analysis

Theorem

$$T_w(z;q) = \sum_{h\geq 0} z^h \left(\sum_{k=0}^{\lfloor wh/2 \rfloor} a_{w[h]}^{(wh-k,k)} q^k \right)$$ has a rational generating function

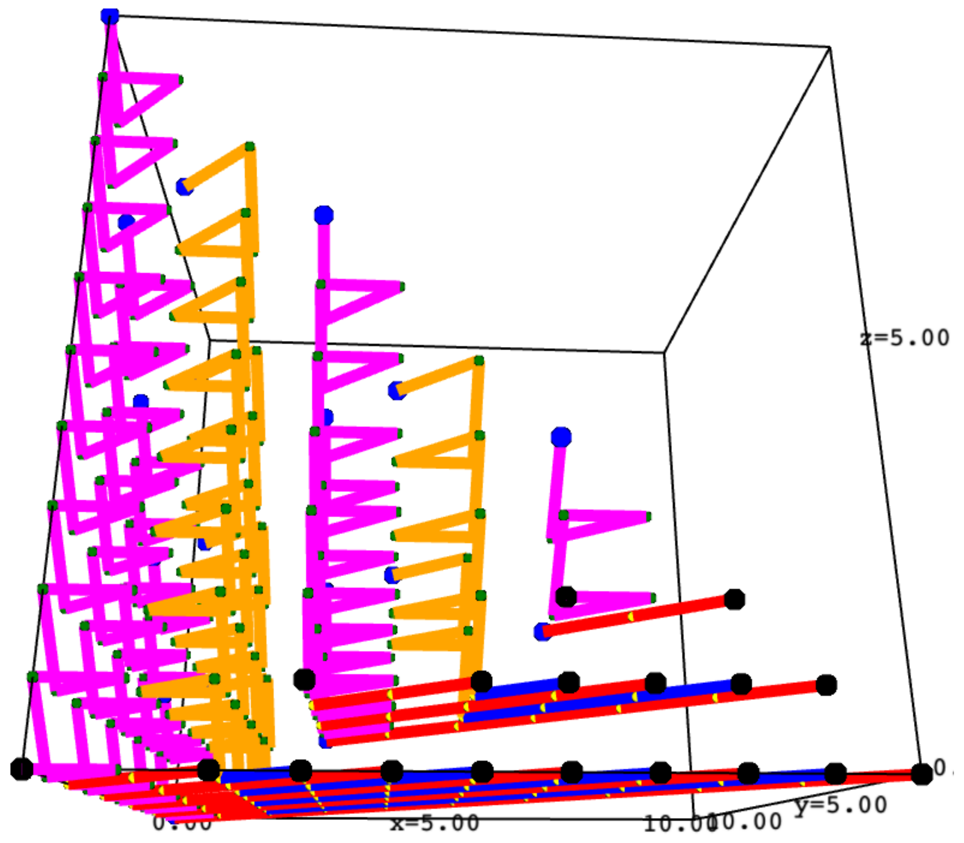
Examples:

$$T_2 = \frac{1}{(1-z)(1-q^2z^2)}$$

$$T_3 = \frac{1+q^3z^3}{(1-z)(1-q^2z^2)(1-q^6z^4)}$$

$$T_4 = \frac{1+q^3z^3}{(1-z)(1-q^2z^2)(1-q^4z^2)(1-q^6z^3)}$$

Examples:



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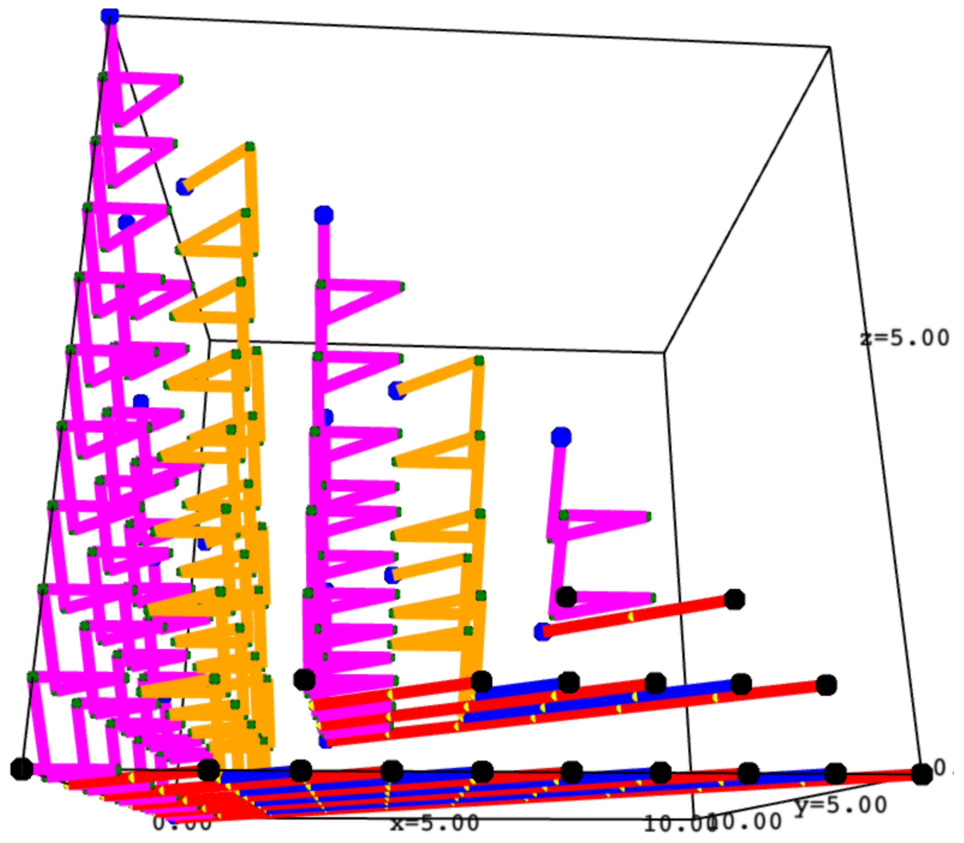
Examples:



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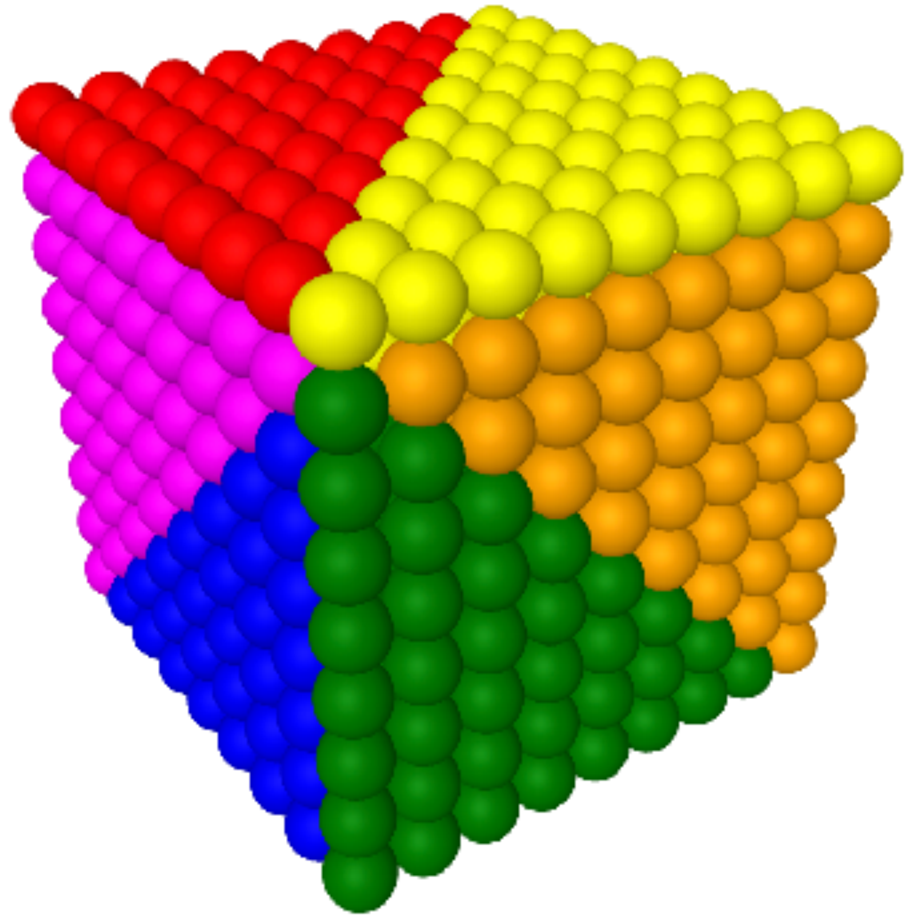
$$T_4 = \frac{1+q^3z^3}{(1-z)(1-q^2z^2)(1-q^4z^2)(1-q^6z^3)}$$

$$T_5 = \frac{\begin{aligned}(-1)\,(q^{34}z^{15} - q^{33}z^{14} - q^{32}z^{14} + q^{32}z^{13} + q^{31}z^{13} + q^{30}z^{13} - q^{30}z^{12} \\ + q^{26}z^{11} - q^{24}z^{10} + q^{23}z^{10} + q^{22}z^{10} - q^{21}z^9 - q^{19}z^9 + q^{19}z^8 + q^{18}z^8 \\ + 2q^{17}z^8 - 2q^{17}z^7 - q^{16}z^7 - q^{15}z^7 + q^{15}z^6 + q^{13}z^6 - q^{12}z^5 - q^{11}z^5 \\ + q^{10}z^5 - q^8z^4 + q^4z^3 - q^4z^2 - q^3z^2 - q^2z^2 + q^2z + qz - 1)\end{aligned}}{(z-1)(qz-1)(q^2z-1)(q^5z^2+1)^2(q^5z^2-1)^3(q^{10}z^4+1)(q^{10}z^4+q^5z^2+1)}$$

Other plethysms:



$$(s_1[s_h])^w(1,q) = \sum_{T \in \mathrm{SYT}^\mu} s_{shape(T)}[s_h](1,q)$$

$$Ehr_{[0,1]^w}(z;q) = \sum_{h \geq 0} (s_1[s_h])^w(1,q) z^h = \frac{\sum_{\pi \in \mathfrak{S}_w} q^{maj(\pi)} z^{des(\pi)}}{(1-z)(1-qz)\cdots(1-q^w z)}$$

Carlitz identity due to MacMahon

Theorem

$$T_\mu(z;q) = \sum_{h \geq 0} z^h \left(\sum_{k \geq 0}^{\lfloor wh/2 \rfloor} a_{\mu[h]}^{(wh-k,k)} q^k \right)$$

has a rational generating function