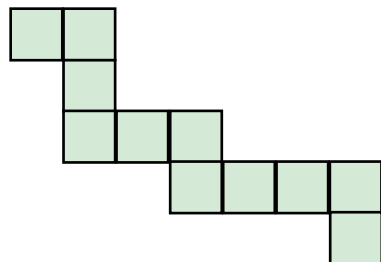


Ribbons and Column Strict Tableaux

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York University



4	5			
3	3	4	5	
2	2	2	3	
1	1	1	1	2

The Symmetric Functions

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \dots]$$
$$\deg(h_k) = k$$

The space of symmetric functions is generated algebraically by the simple homogeneous symmetric functions.

The Schur Functions

$$s_\lambda = \det |h_{\lambda_i + i - j}|$$

Example:

$$s_{(2,2,1)} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = h_2^2 h_1 - h_2 h_3 - h_3 h_1^2 + h_4 h_1$$

$$h_3^2 h_7 h_2 h_1^3 h_4$$

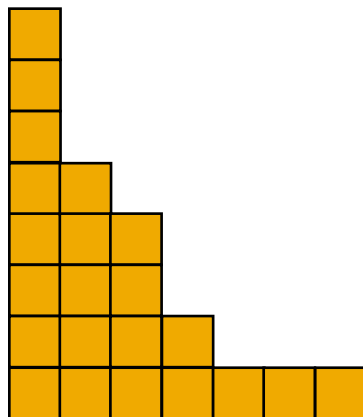
$$\text{degree } 2 \cdot 3 + 7 + 2 + 3 \cdot 1 + 4 = 22$$

$$h_7 h_4 h_3^2 h_2 h_1^3$$

$$h_{(7,4,3,3,2,1,1,1)}$$

definition: a *partition* of n
sequence of non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$
such that $\lambda_1 + \lambda_2 + \dots + \lambda_{\ell(\lambda)} = n$
and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)} > 0$.

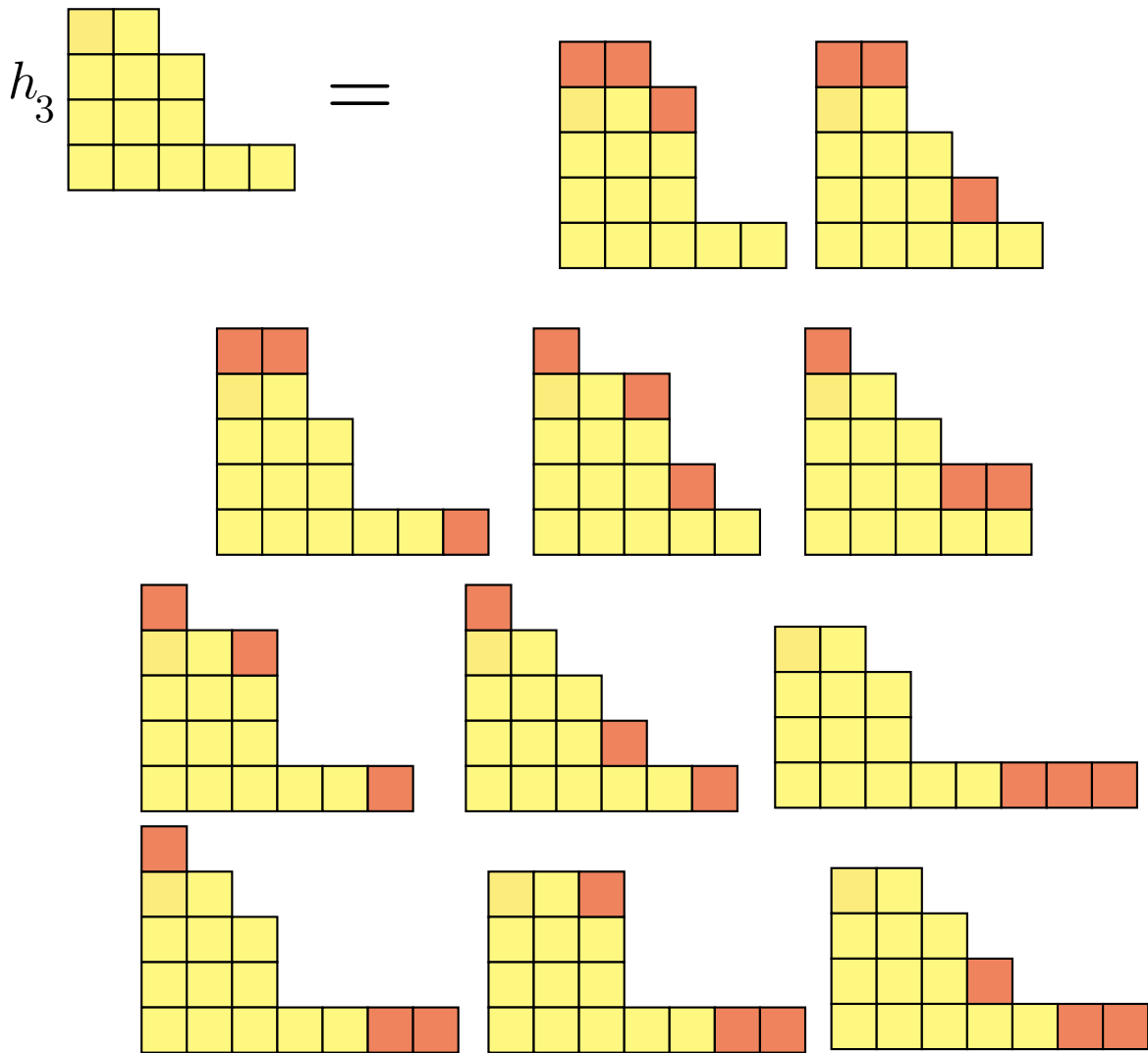
Young diagram of a partition



The Pieri Rule

$$h_m s_\lambda = \sum_{\mu} s_\mu$$

μ contains λ and the difference has at most one cell per column



Homogeneous \rightarrow Schur

$$h_{\begin{array}{|c|c|c|c|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} = h_5 h_4 h_1$$

$$h_5 = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} = s_5$$

$$\begin{aligned} h_4 h_5 &= \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline 2 & 2 & 2 & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 2 & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{array} \\ &+ \begin{array}{|c|c|c|c|c|c|c|c|} \hline 2 & & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{array} \\ &= s_9 + s_{8,1} + s_{7,2} + s_{6,3} + s_{5,4} \end{aligned}$$

$$\begin{aligned} h_1 h_4 h_5 &= \begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline 2 & 2 & 2 & 2 & \\ \hline 1 & 1 & 1 & 1 & 1 \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 3 \\ \hline 2 & 2 & 2 & 2 & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 3 \\ \hline 2 & 2 & 2 & 2 & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array} + \\ &+ \begin{array}{|c|c|c|c|c|c|} \hline 3 & & & & & \\ \hline 2 & 2 & 2 & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 3 \\ \hline 2 & 2 & 2 & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & 3 \\ \hline 2 & 2 & 2 & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{array} \\ &+ \begin{array}{|c|c|c|c|c|c|c|} \hline 3 & & & & & & \\ \hline 2 & 2 & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & 3 \\ \hline 2 & 2 & 3 & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & 3 \\ \hline 2 & 2 & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{array} \\ &+ \begin{array}{|c|c|c|c|c|c|c|c|} \hline 3 & & & & & & & \\ \hline 2 & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & 3 \\ \hline 2 & 3 & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & & & & & & & 2 \\ \hline & & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} \\ &+ \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 3 & & & & & & & & \\ \hline & & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & & & & & & & 3 \\ \hline & & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} \\ &= s_{10} + 2s_{9,1} + 2s_{8,2} + s_{8,1,1} + 2s_{7,3} + s_{7,2,1} \\ &\quad + 2s_{6,4} + s_{6,3,1} + s_{5,5} + s_{5,4,1} \end{aligned}$$

The Homogeneous functions

$$h_\lambda = \sum_T s_{\lambda(T)}$$

Example:

$$h_{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}} =$$

$$\begin{array}{|c|c|} \hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 2 & 2 & & \\ \hline 1 & 1 & 3 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 2 & 3 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 2 & & & \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 2 & 3 & & \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|c|} \hline 3 & 3 & & \\ \hline 1 & 1 & 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline 1 & 1 & 2 & 2 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 3 \\ \hline \end{array}$$

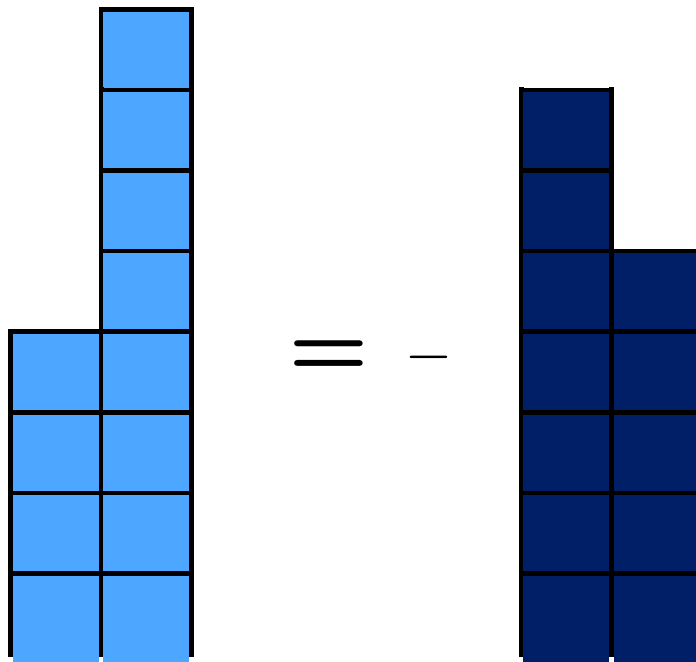
$$= s_{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}} + 2s_{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}} + 3s_{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}}$$

$$+ s_{\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}} + 2s_{\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array}}$$

For each column strict tableau of content λ there is exactly one Schur function that appears in the sum.

Rule 1: A Straightening Rule for Schur Functions

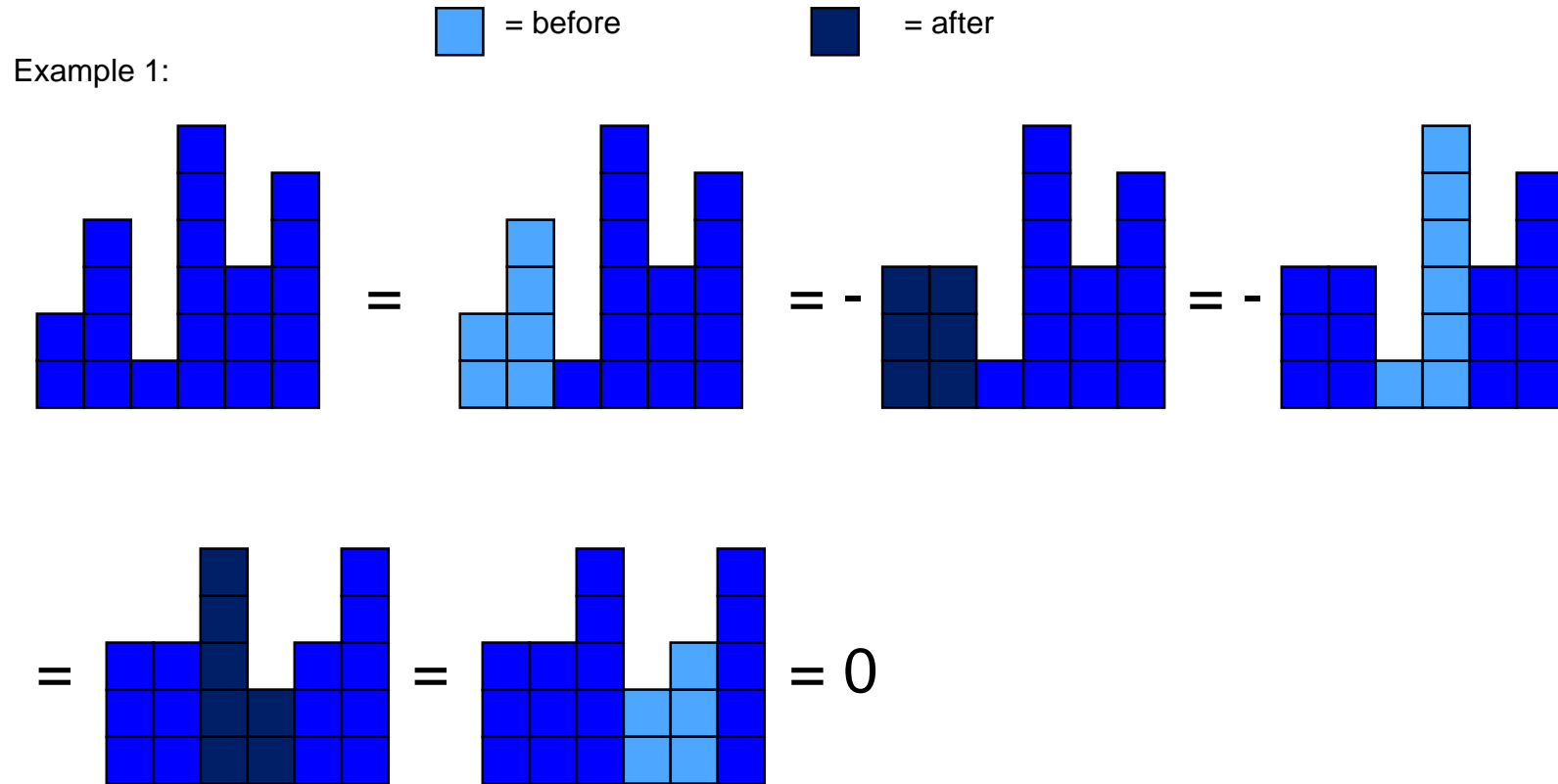
A column of size m & a column of size $n =$
 – a col. of size $n - 1$ & a col. of size $m + 1$



Note: a column of size m on a column of $m + 1$

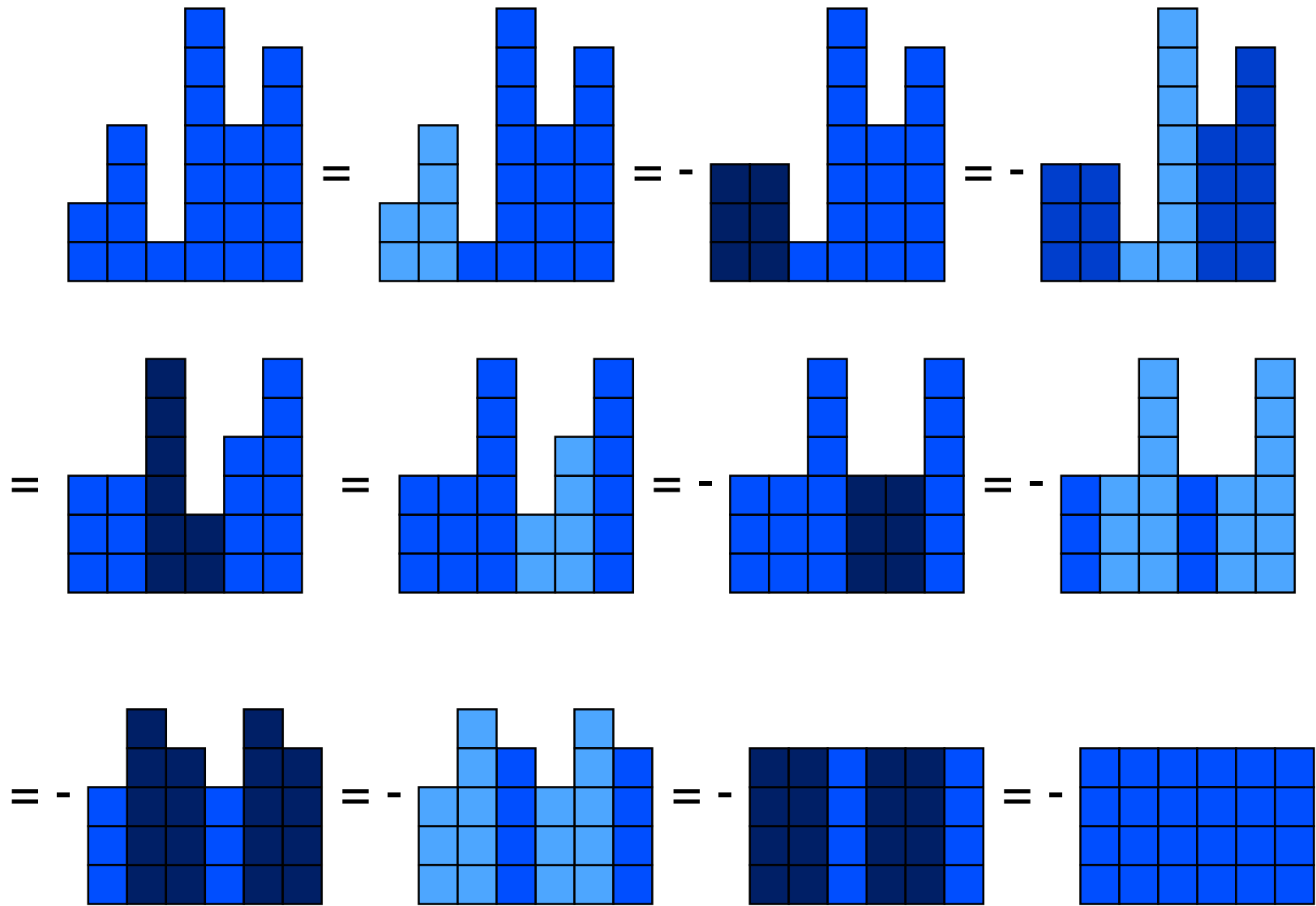
The diagram shows the equation: a light blue Young diagram with two columns of heights m and $m+1$ equals minus one times a dark blue Young diagram with two columns of heights $m+1$ and m , which equals zero.

An example of the straightening rule:



Example 2:

 = before  = after



Rule 2: The Littlewood-Richardson Rule

A combinatorial rule for expanding skew Schur functions in terms of Schur functions indexed by partitions.

Definition: skew-Schur function
for λ/μ skew partition

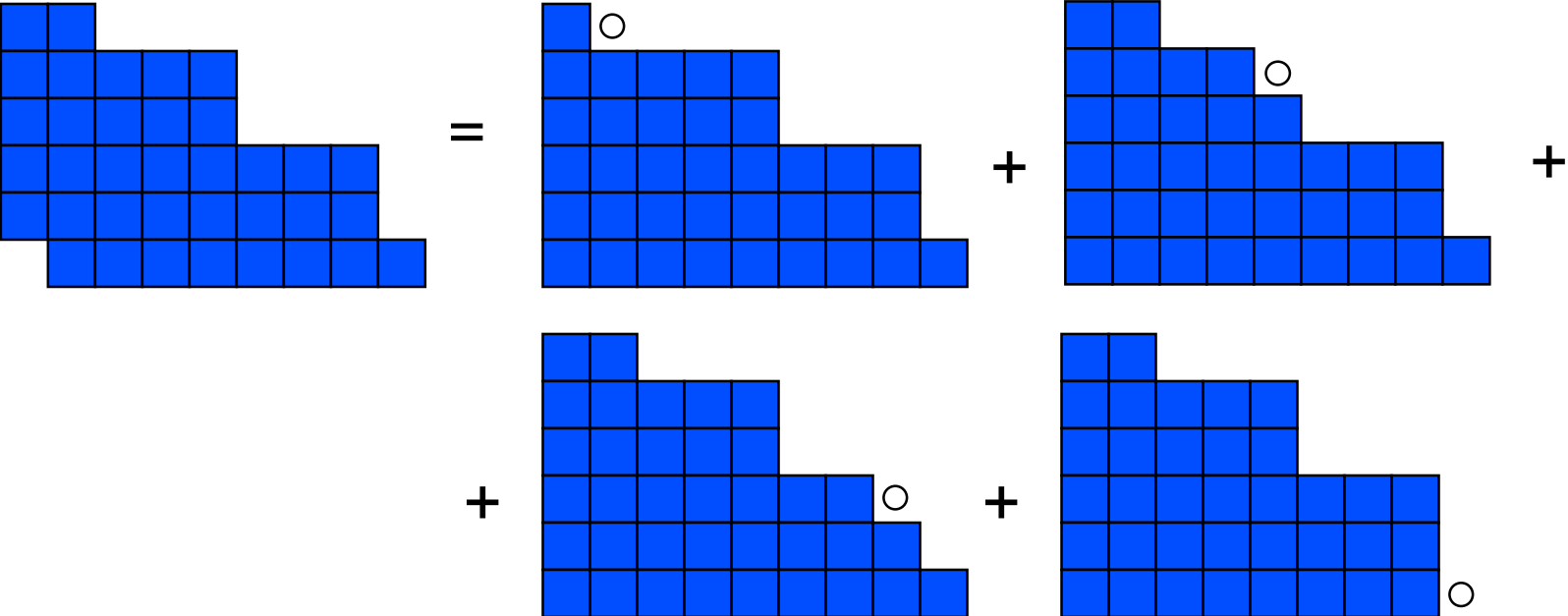
$$s_{\lambda/\mu} = \det |h_{\lambda_i - \mu_j + i - j}|$$

The LR-rule:

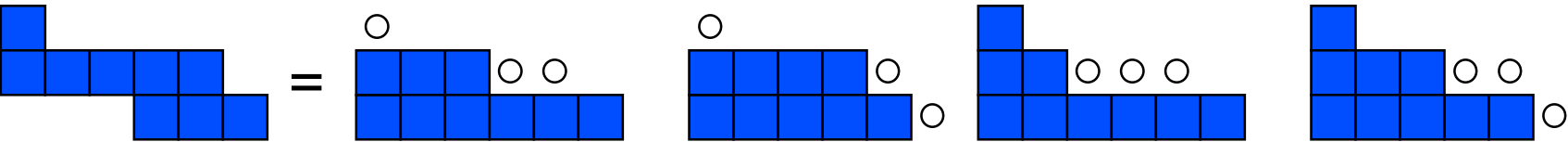
$$s_{\lambda/\mu} = \sum_{\nu} c_{\nu\mu}^{\lambda} s_{\nu}$$

where the coefficients $c_{\nu\mu}^{\lambda}$ are the number of ways of filling a Young diagram of shape λ/μ with ν_1 1's, ν_2 2's, ν_3 3's, etc. such that the filling increases weakly in the rows, strictly in the columns AND the for each k , the first k entries of the reverse reading word has partition content.

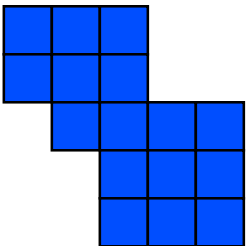
Example 1: In the case when the inner partition consists of only one square the result is equivalent to removing each of the corner cells of the outer partition:



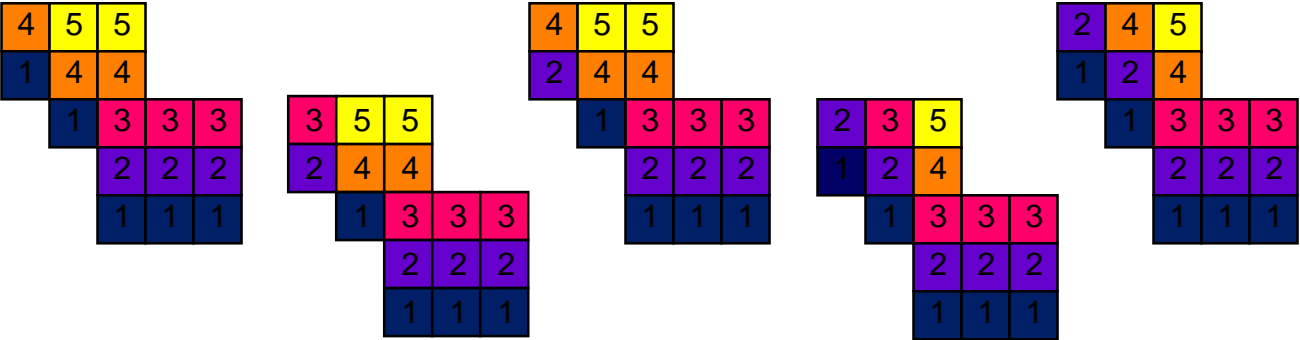
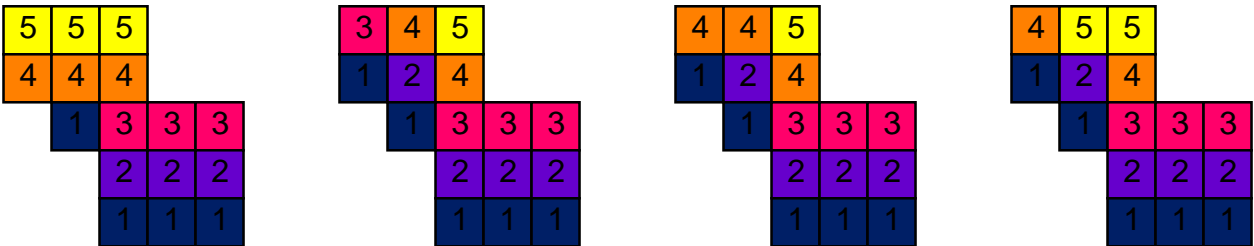
Example 2: In the case that the inner partition is a single row, the result is equivalent to removing all horizontal strips of the same size from the border of the outer partition.



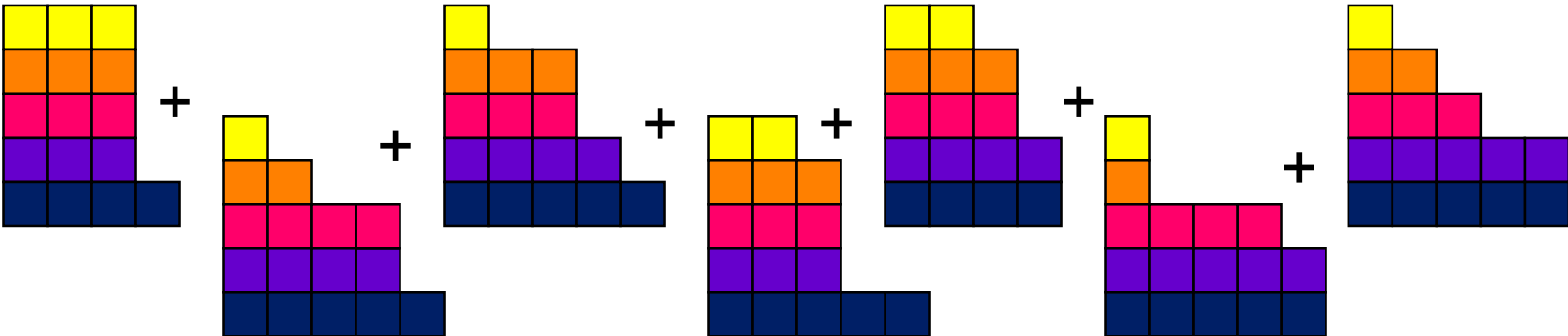
Example 3: Something a little more complicated



=

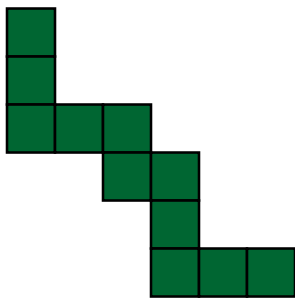


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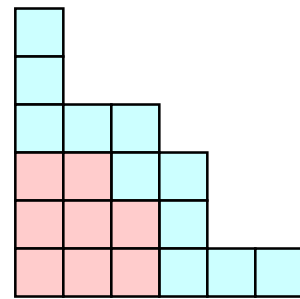


Ribbon Operators

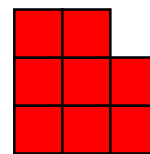
Ribbon operators use a combination of the operation of straightening columns followed by the Littlewood-Richardson rule.



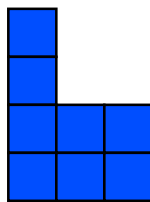
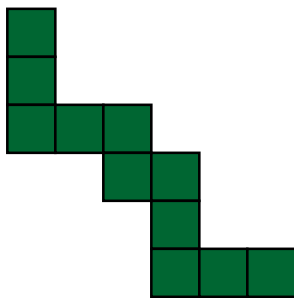
means first add these columns
on the left



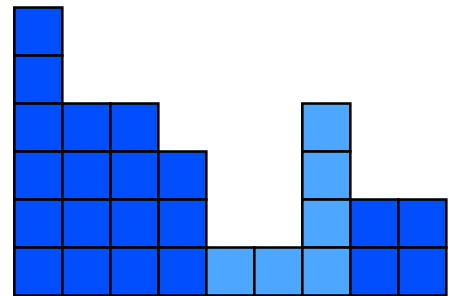
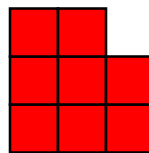
then remove the shape



Example 1:

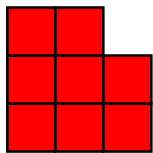


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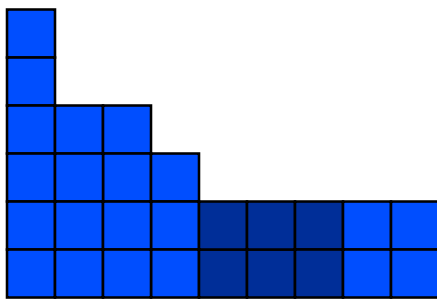


remove

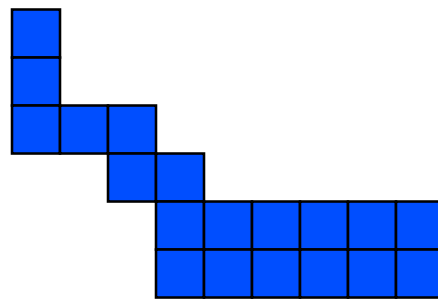
=



remove



=



Now reduce this with the Littlewood-Richardson rule.

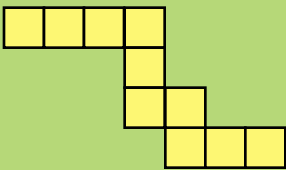
A ribbon may be represented by a sequence of rows.
 $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$ with $\alpha_i > 0$ for $1 \leq i \leq \ell(\alpha)$

$D(\alpha) =$ descent set of α

$$= \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)-1}\}$$

Example

$\alpha = (4, 1, 2, 3)$



$D(\alpha) = \{4, 5, 7\}$

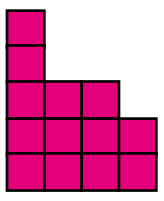
We will identify a ribbon operator with the descent set
 If $D \subseteq \{1, 2, \dots, m-1\}$, then S^D is the ribbon operator
 whose associated ribbon α has $D(\alpha) = D$

Theorem (Zabrocki):

$$H_{1^m} := \sum_{D \subseteq \{1, 2, \dots, m-1\}} S^D$$


$$H_{1^m}(h_\lambda) = h_{\lambda + (1^m)}$$



Example


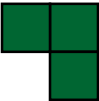


$$H_{1^5} H_{1^3} H_{1^3} H_{1^2} 1 = h_{(1, 1, 3, 4, 4)} = h$$




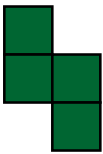
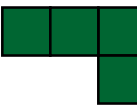
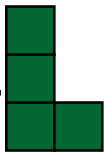
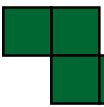

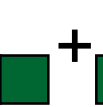

Theorem: a dual Pieri rule (Zabrocki)

The sum of all ribbon operators of size m adds a column on the homogeneous symmetric functions.

 adds a column of size 1 on a homogeneous symmetric function with at most 1 part

 +  adds a column of size 2 on a homogeneous symmetric function with at most 2 parts

 +  +  +  adds a column of size 3 on a homogeneous symmetric function with at most 3 parts

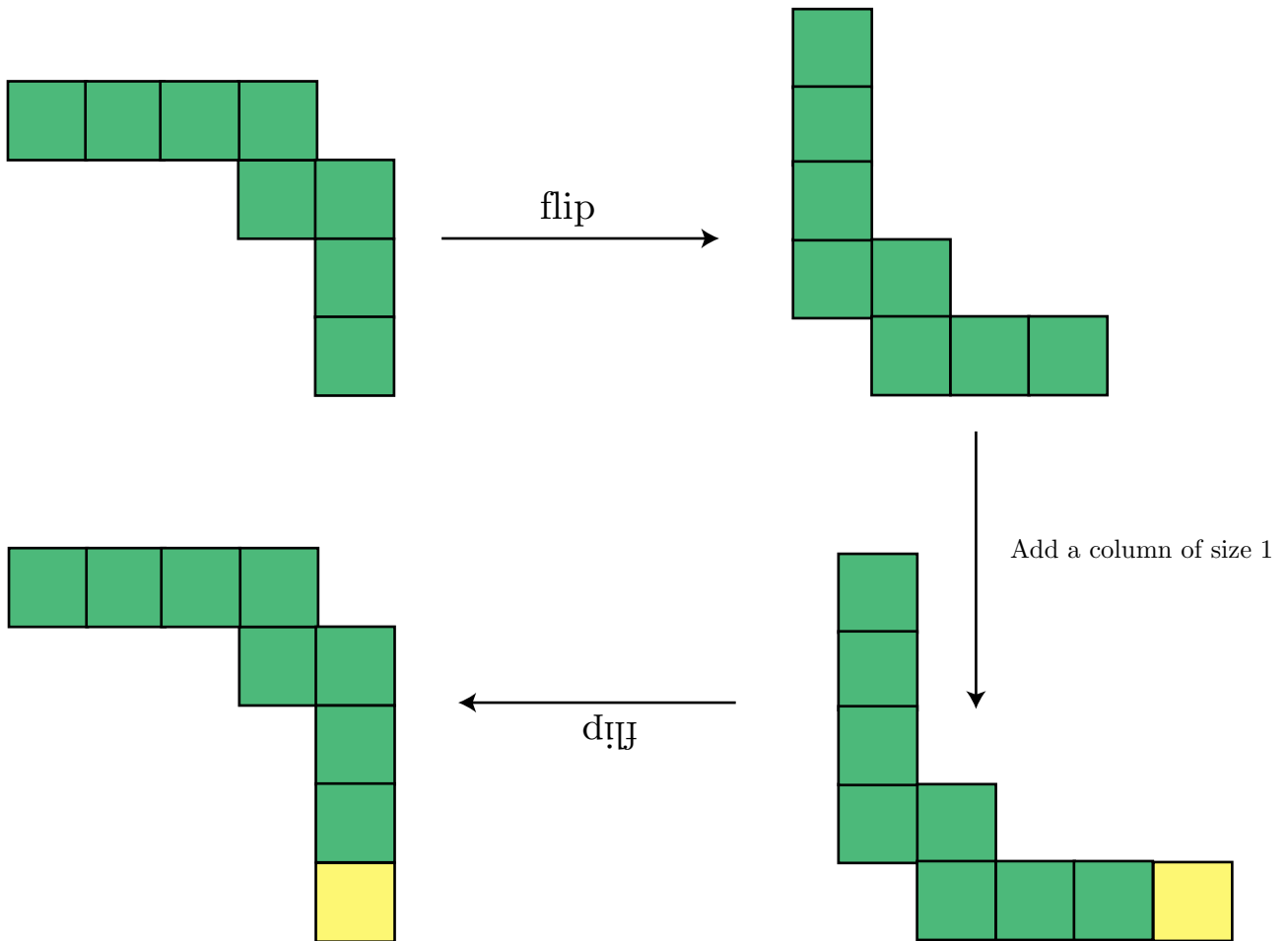
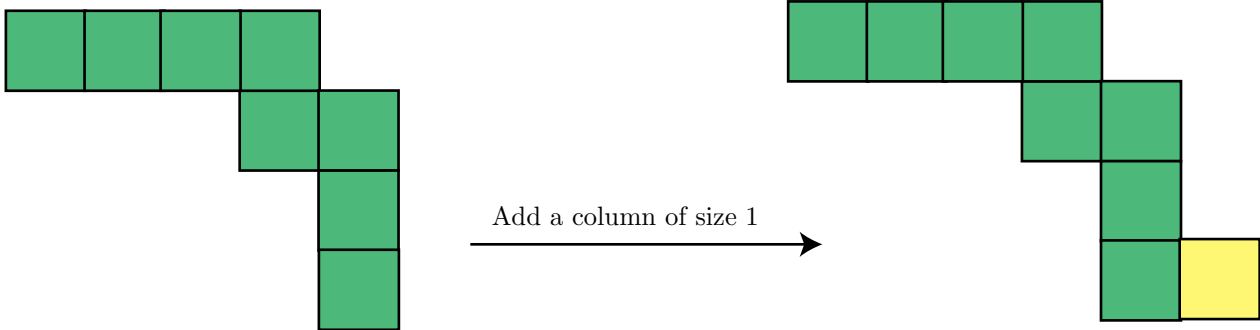
 +  +  +  +  +  +  +  + 

adds a column of size 4... etc.

Example 1: adding a column of size 3 on the empty Schur function yields h

$$\begin{aligned}
 & \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right) = \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right) \\
 & = \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) + 2 \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right) = h_{\left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right)}
 \end{aligned}$$

Building ribbon operators



A q -refinement

Theorem:

$$H_{1^m}^q := \sum_{D \subseteq \{1, 2, \dots, m-1\}} q^{\sum_{i \in D} i} S^D$$

$$H_{1^m}^q H_\mu[X; q] = H_{\mu + (1^m)}[X; q]$$

$H_\mu[X; q] \Big|_{s_\lambda} = q$ count column strict tableaux shape λ content μ
 $=$ Kostka Foulkes polynomial

$$H_{(2,2,2)}[X; q] = s_6 + q(1+q)s_{5,1} + q^2(q^2+q+1)s_{4,2} + q^3s_{4,1,1} + q^3s_{3,3} + q^4(1+q)s_{3,2,1} + q^6s_{2,2,2}$$

A qt -refinement

Theorem:

$$H_{1^m}^{qt} := \sum_{D \subseteq \{1, 2, \dots, m-1\}} q^{\sum_{i \in D} i} (S^D)^t$$

$$H_{1^m}^{qt} H_\mu[X; q, t] = H_{\mu + (1^m)}[X; q, t]$$

$H_\mu[X; q, t] \Big|_{s_\lambda} = q, t$ count column strict tableaux shape λ content $(1^{|\mu|})$
 $=$ Macdonald-Kostka polynomial

Some examples

Diagram illustrating the decomposition of a Young diagram (left) into two Young diagrams (right). The left diagram consists of teal squares arranged in a shape with a height of 3. The right diagram consists of blue squares arranged in a shape with a height of 2.

$$\begin{array}{c}
 \square \square \\
 \square \square \\
 \square \square
 \end{array}
 +
 \begin{array}{c}
 \square \square \\
 \square \square
 \end{array}
 =
 \begin{array}{c}
 \square \square \square \square \square \\
 \square \square \square \square \square \square \square \square
 \end{array}$$

Diagram illustrating the decomposition of a Young diagram (left) into two Young diagrams (right). The left diagram consists of teal squares arranged in a shape with a height of 2. The right diagram consists of blue squares arranged in a shape with a height of 2.

$$\begin{array}{c}
 \square \square \square \square \\
 \square \square \square \square
 \end{array}
 +
 \begin{array}{c}
 \square \square \square \square \\
 \square \square \square \square \square \square \square \square
 \end{array}$$

Diagram illustrating a Young diagram (left) equal to zero. The left diagram consists of teal squares arranged in a shape with a height of 3. The right side is labeled $= 0$.

$$\begin{array}{c}
 \square \square \\
 \square \square \\
 \square \square
 \end{array}
 = 0$$

Diagram illustrating a Young diagram (left) equal to a negative sum of three Young diagrams (right). The left diagram consists of teal squares arranged in a shape with a height of 3. The right side shows three blue Young diagrams, each preceded by a minus sign and the number 3.

$$\begin{array}{c}
 \square \square \\
 \square \square \\
 \square \square
 \end{array}
 = -
 \begin{array}{c}
 \square \square \square \square \\
 \square \square \square \square \square \square \square \square
 \end{array}
 - 3
 \begin{array}{c}
 \square \square \square \square \square \\
 \square \square \square \square \square \square \square \square
 \end{array}
 - 3
 \begin{array}{c}
 \square \square \square \square \square \square \\
 \square \square \square \square \square \square \square \square
 \end{array}$$