Ribbons and Homogeneous Symmetric Functions

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The Symmetric Functions

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots]$$

The space of symmetric functions is generated algebraically by the simple homogeneous symmetric functions. This may be taken as a definition.

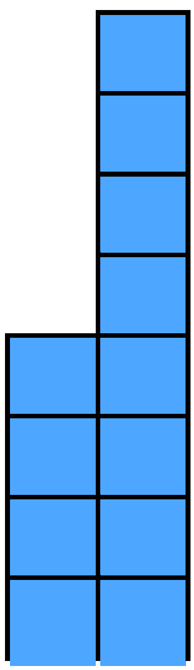
The Schur Functions

$$s_\lambda = det|h_{\lambda_i + i - j}|$$

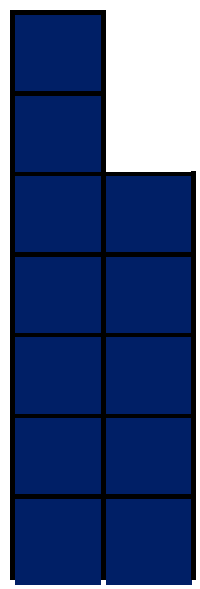
The definition of the Schur polynomials is well known and they are a fundamental basis of the symmetric functions. Schur functions will be identified here with the Young diagrams for the partition.

Rule 1:
A Straightening Rule for Schur Functions

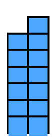
A column of size m & a column of n =
− a col. of size $n-1$ & a col. of size $m+1$



$= -$



Note: a column of size m on a column of $m+1$



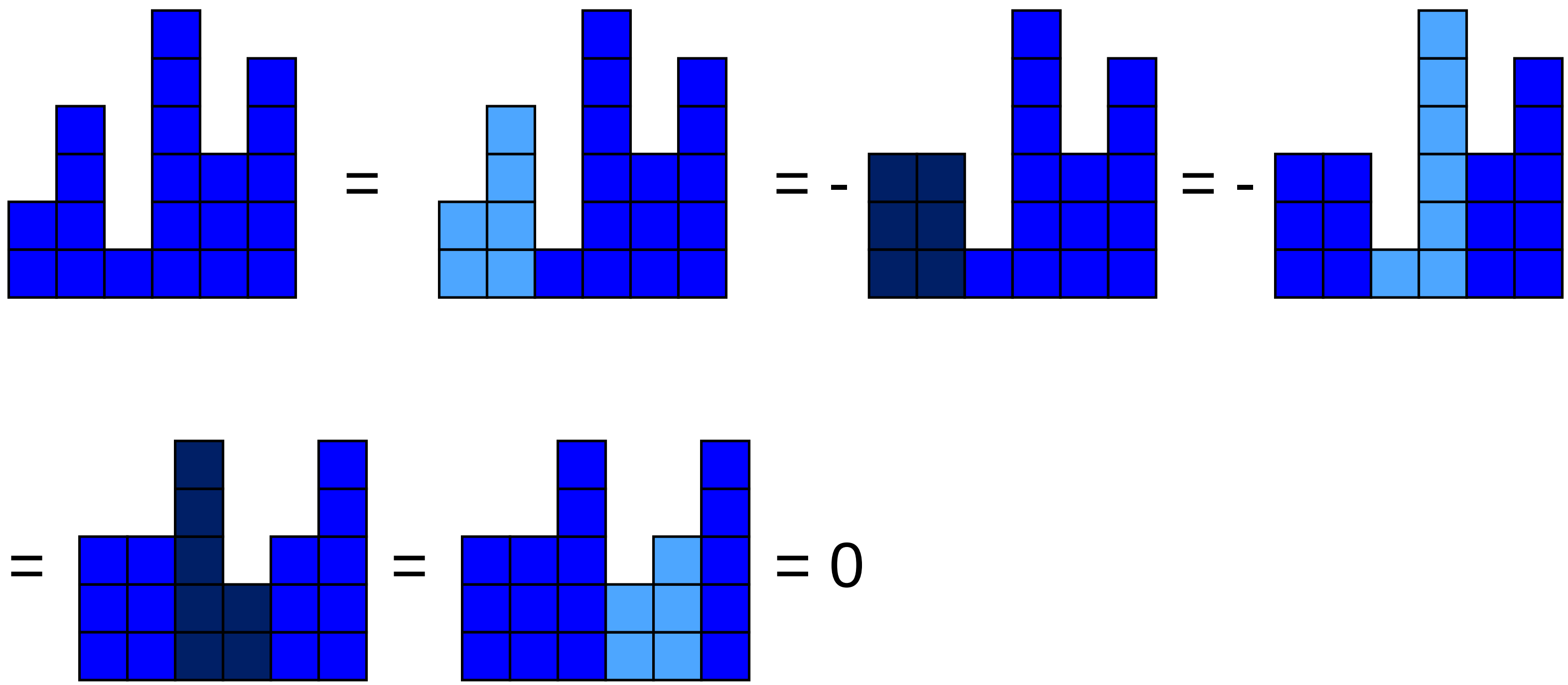
$= -$

$= 0$

An example of the straightening rule:

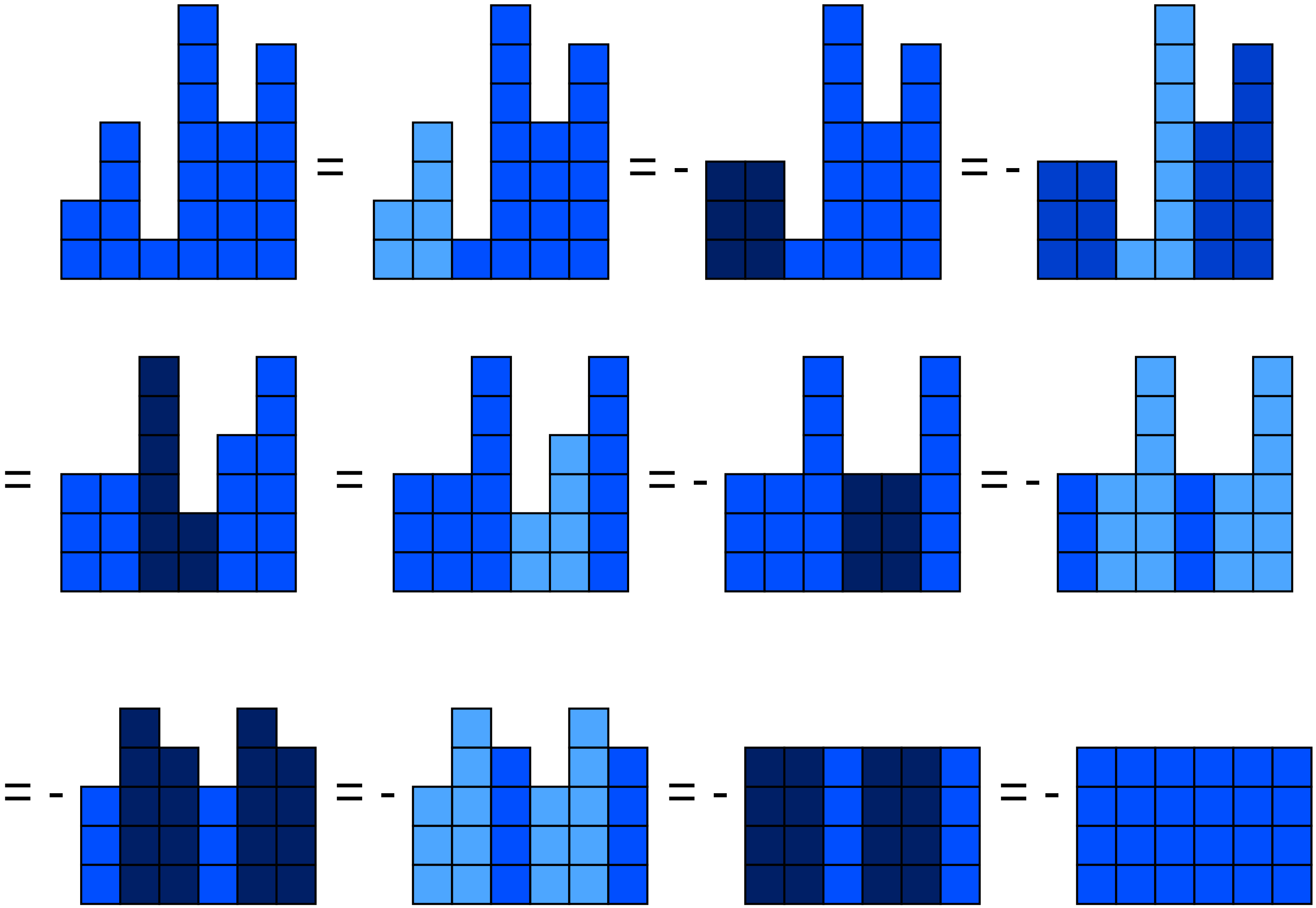
= before   = after

Example 1:



= before = after

Example 2:



Rule 2: The Littlewood-Richardson Rule

A combinatorial rule for expanding skew Schur functions in terms of Schur functions indexed by partitions.

Definition: skew-Schur function for λ/μ skew partition

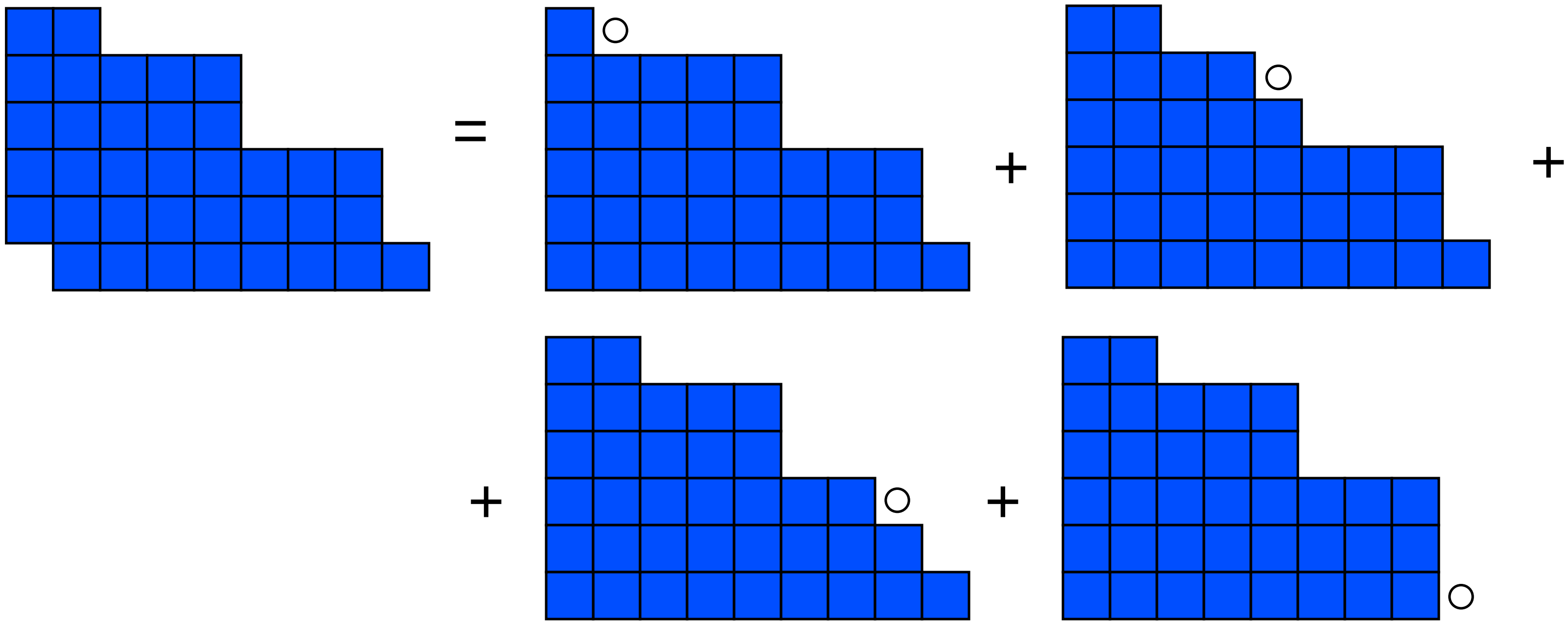
$$s_{\lambda/\mu} = det|h_{\lambda_i - \mu_j + i - j}|$$

The LR-rule:

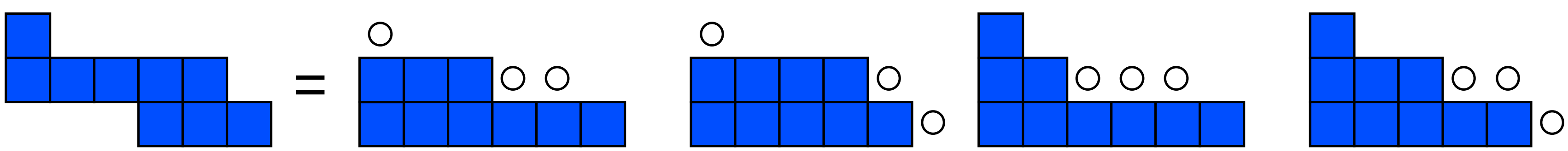
$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\nu\mu} s_{\nu}$$

where the coefficients $c^{\lambda}_{\nu\mu}$ are the number of ways of filling a Young diagram of shape λ/μ with ν_1 1's, ν_2 2's, ν_3 3's, etc. such that the filling increases weakly in the rows, strictly in the columns AND the for each k, the first k entries of the reverse reading word has partition content.

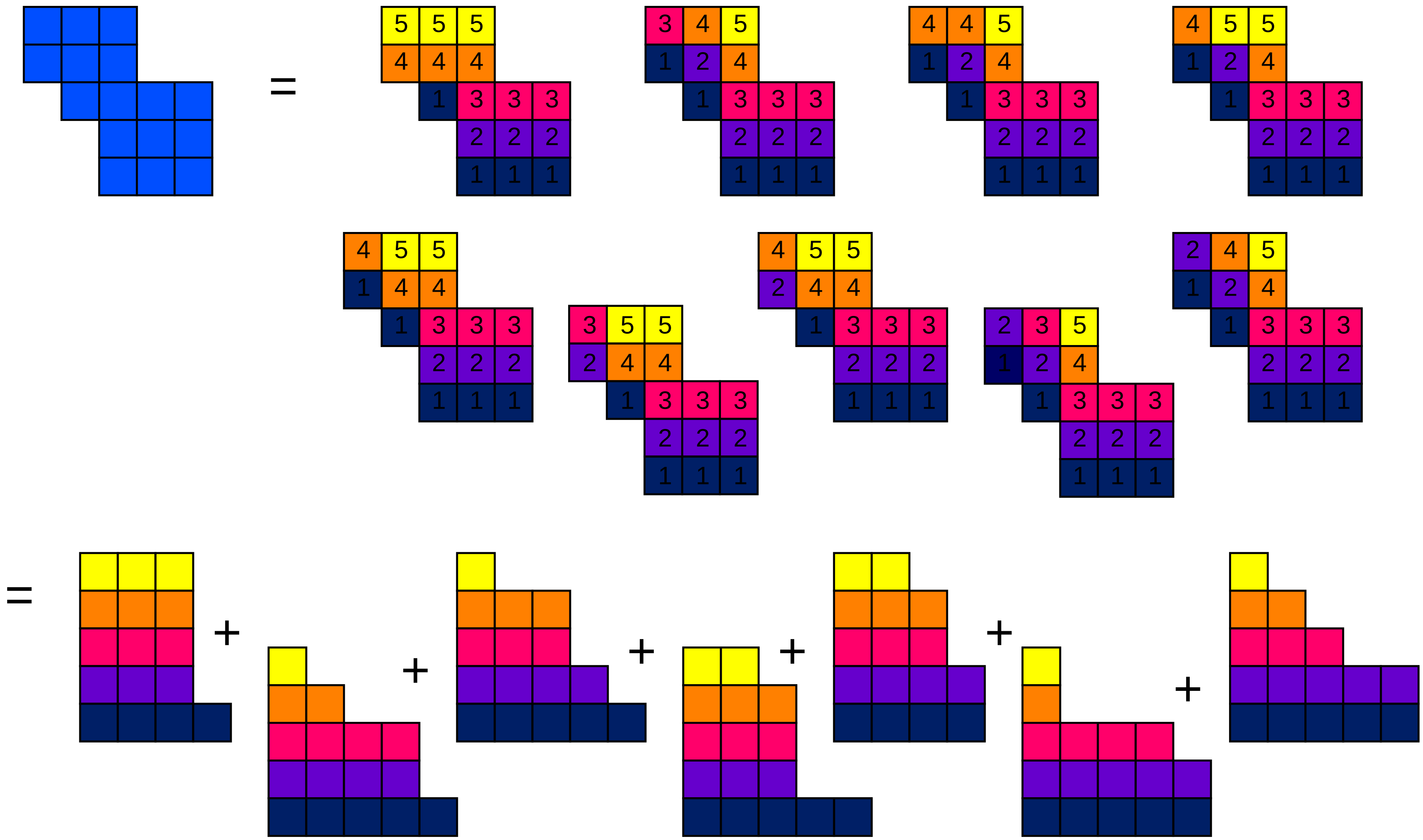
Example 1: In the case when the inner partition consists of only one square the result is equivalent to removing each of the corner cells of the outer partition:



Example 2: In the case that the inner partition is a is a single row, the result is equivalent to removing all horizontal strips of the same size from the border of the outer partition.

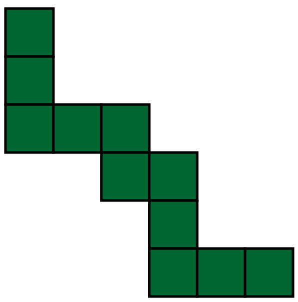


Example 3: Something a little more complicated

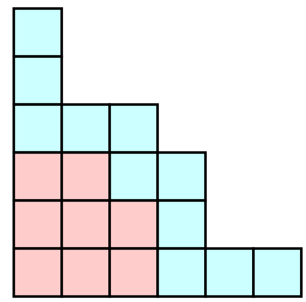


Ribbon Operators

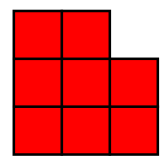
Ribbon operators use a combination of the operation of straightening columns followed by the Littlewood-Richardson rule.



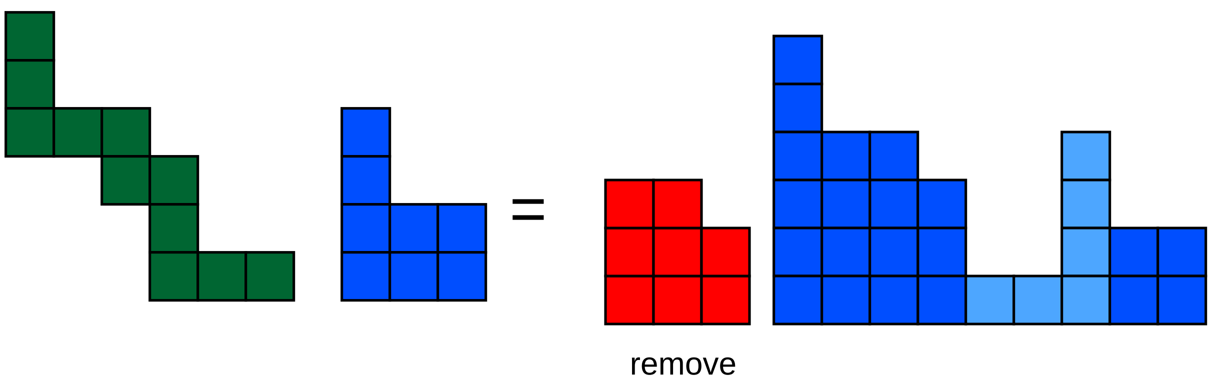
means first add these columns on the left

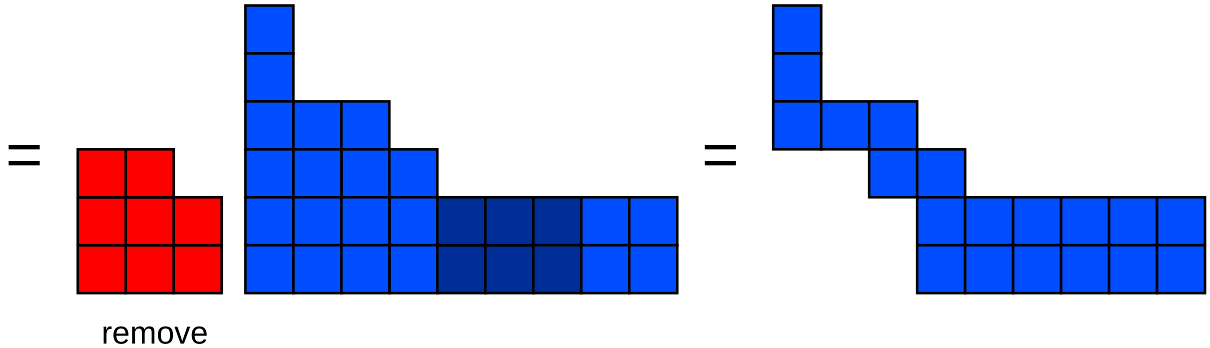


then remove the shape



Example 1:

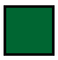




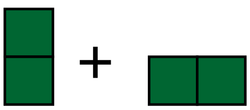
Now reduce this with the Littlewood-Richardson rule.

A dual Pieri rule :

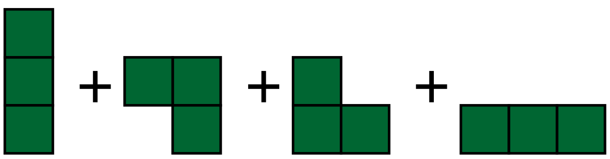
The sum of all ribbon operators of size m adds a column on the homogeneous symmetric functions.



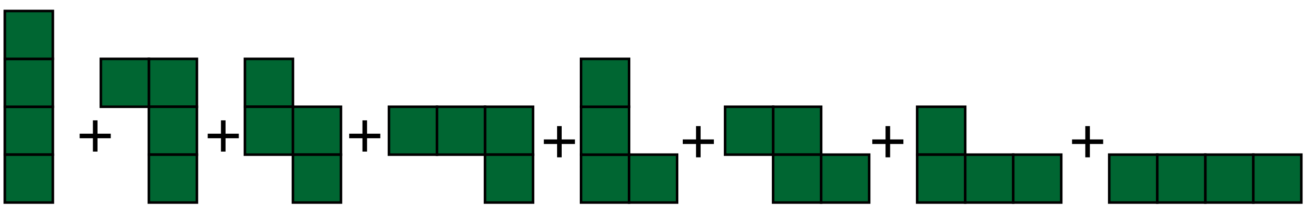
adds a column of size 1 on a homogeneous symmetric function with at most 1 part



adds a column of size 2 on a homogeneous symmetric function with at most 2 parts

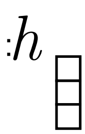


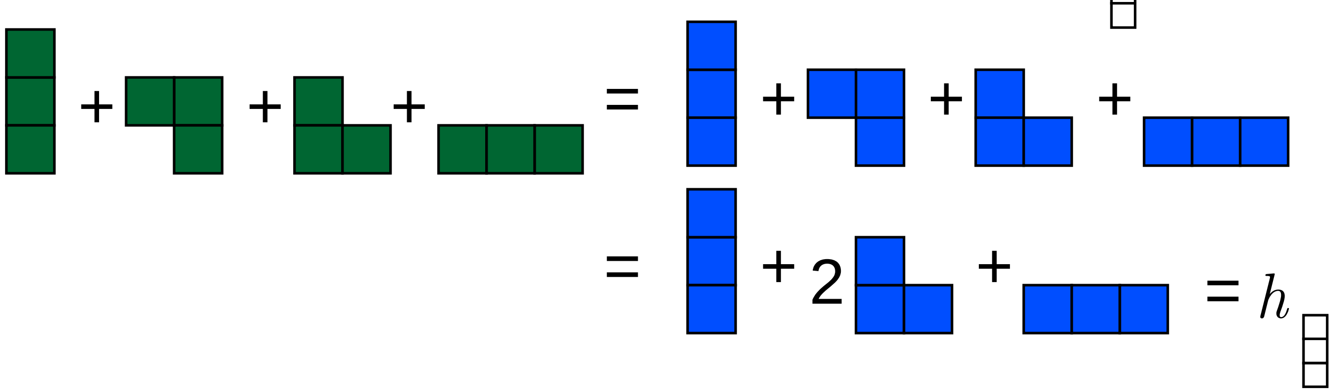
adds a column of size 3 on a homogeneous symmetric function with at most 3 parts



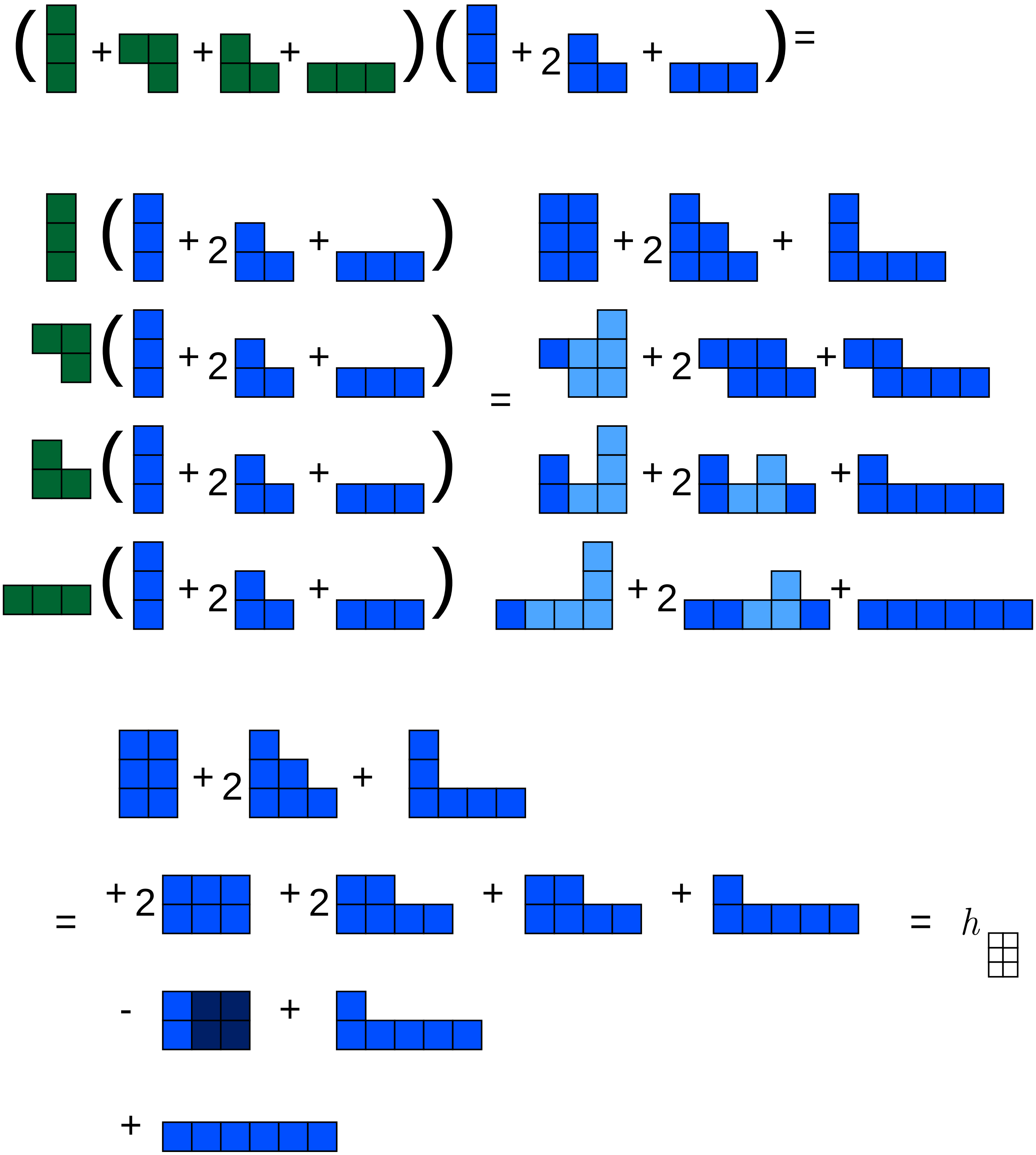
adds a column of size 4... etc.

Example 1: adding a column of size 3 on the empty Schur function yields :





Example 2: Add a column of size 3 on $h_{\square}$ yields $h_{\square}$



Open question:

Combinatorially prove the positivity of a composition of these operators (they yield the homogeneous symmetric functions, of course they are Schur positive). Does this give a new combinatorial interpretation of the homogeneous symmetric functions?

Generalizations:

There exist q (a dual Morris recurrence) and q, t (a Macdonald-Morris recurrence) analogs of the ribbon rule. Can these generalized operators be used to show positivity of the Hall-Littlewood and Macdonald symmetric functions?