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q and qt-analogs of non-commutative symmetric functions

The Symmetric Functions

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots]$$
$$deg(h_k) = k$$

The space of symmetric functions is generated algebraically by the simple homogeneous and power symmetric functions.

The Schur Functions

$$s_\lambda = det|h_{\lambda_i+i-j}|$$

Example:

$$s_{(2,2,1)} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = {h_2}^2 h_1 - h_2 h_3 - h_3 {h_1}^2 + h_4\, h_1$$

More symmetric functions

$$e_k := e_{k-1}h_1 - e_{k-2}h_2 + \cdots + (-1)^k h_k$$

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$$

$$s_\lambda = det|h_{\lambda_i+i-j}|$$

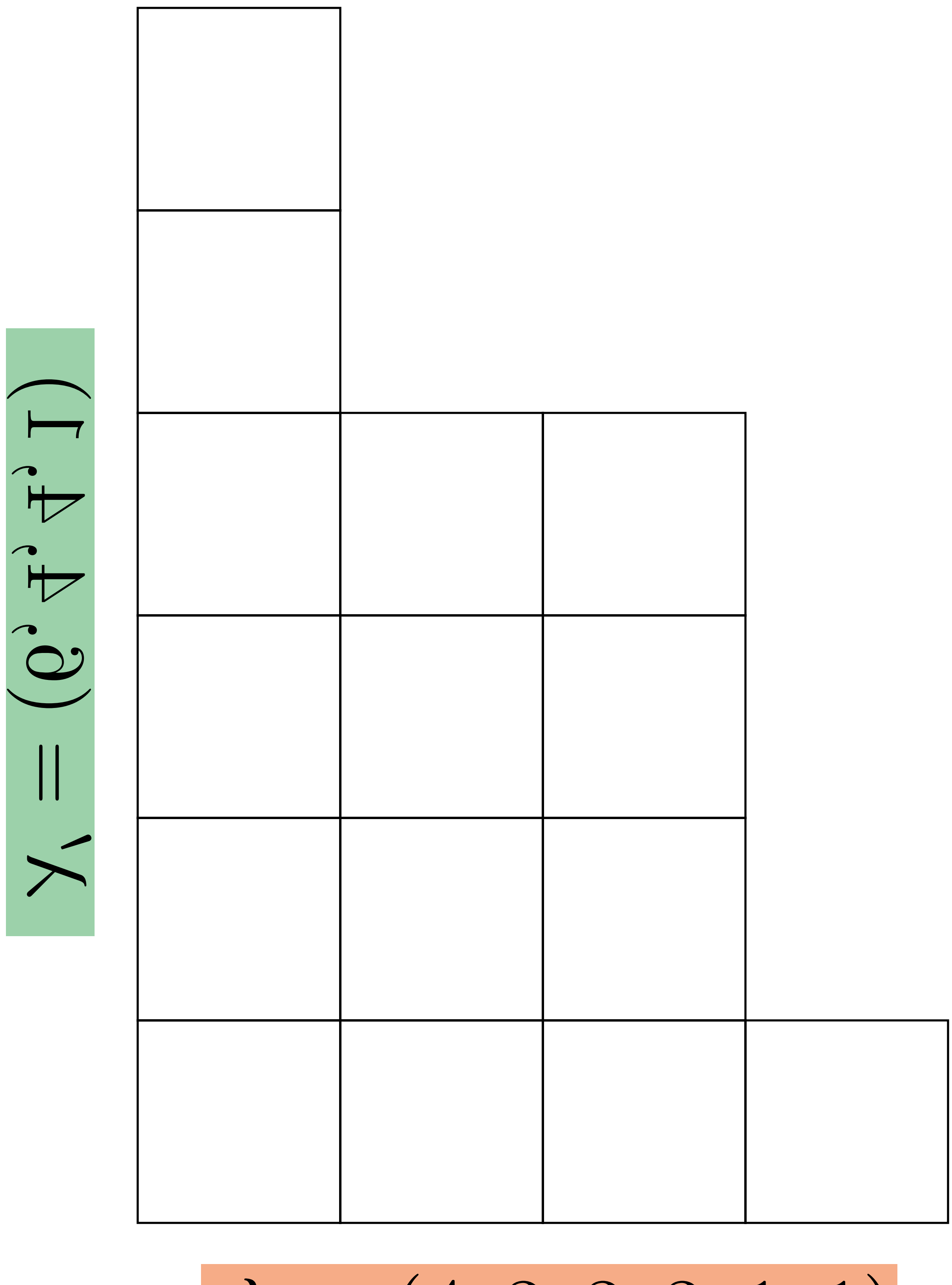
more generally

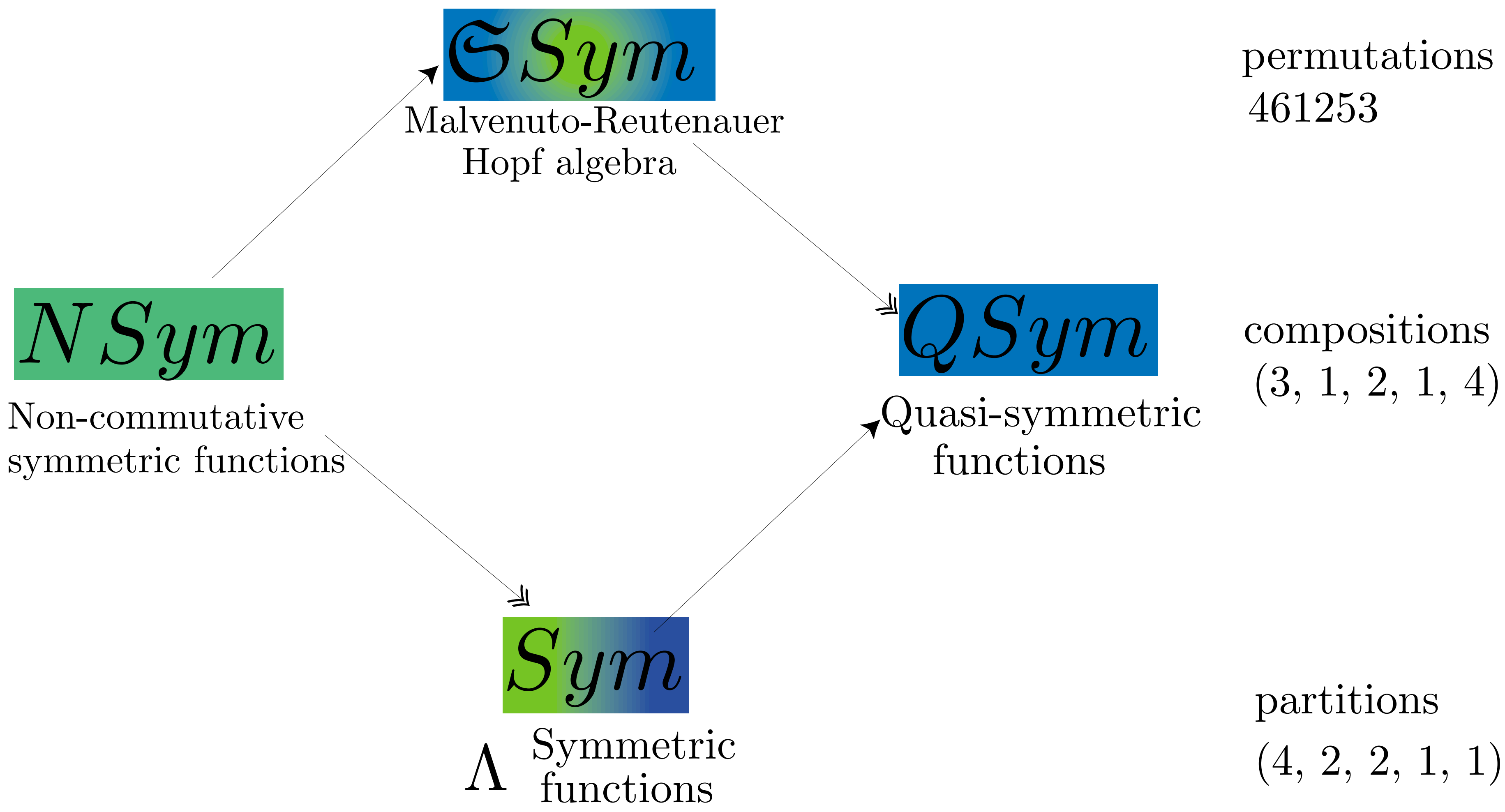
$$s_{\lambda/\mu} = det|h_{\lambda_i-\mu_j+i-j}|$$

$$\{s_\lambda\}_{\lambda\vdash n} \qquad \{h_\lambda\}_{\lambda\vdash n} \qquad \{e_\lambda\}_{\lambda\vdash n}$$

bases for the symmetric functions of degree n

$$\omega(h_\lambda) = e_\lambda \qquad \omega(s_\lambda) = s_{\lambda'}$$


$\lambda' = (6, 4, 4, 1)$
$\lambda = (4, 3, 3, 3, 1, 1)$



𝔖Sym
Malvenuto-Reutenauer
Hopf algebra
permutations
461253
NSym
Non-commutative
symmetric functions
QSym
Quasi-symmetric
functions
compositions
(3, 1, 2, 1, 4)
Sym
Λ Symmetric
functions
partitions
(4, 2, 2, 1, 1)

Non-commutative symmetric functions

$$NSym = \mathbb{Q}\langle \mathtt{h}_1, \mathtt{h}_2, \mathtt{h}_3, \ldots \rangle$$
$$deg(\mathtt{h}_k) = k$$
$$\mathsf{h}_i \mathsf{h}_j \neq \mathsf{h}_j \mathsf{h}_i$$

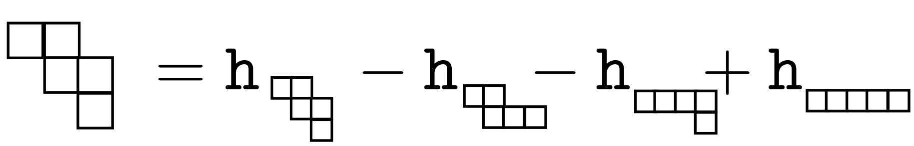
The space of non-commutative symmetric functions is the free algebra with one generator in each degree

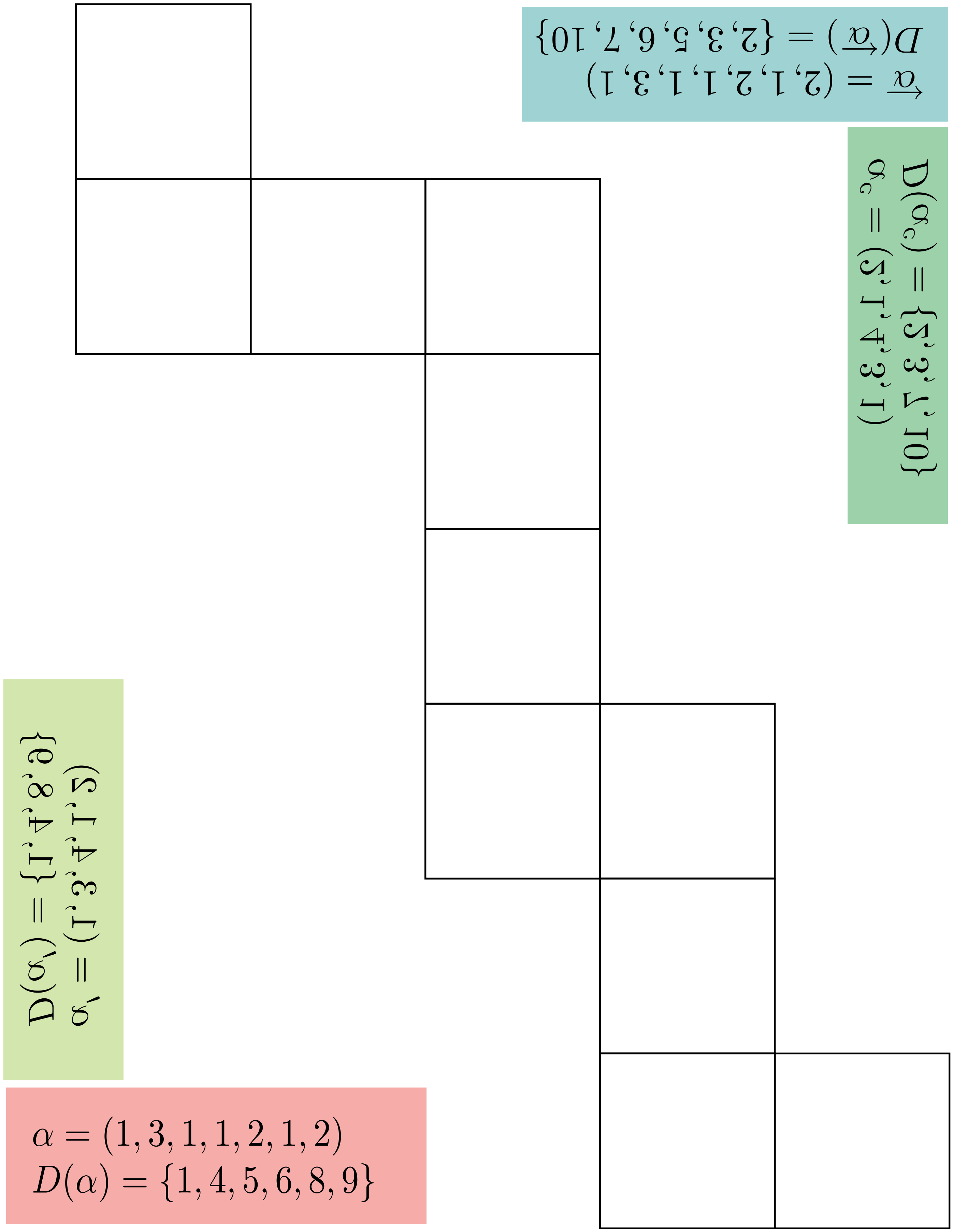
The NCSchur Functions

$$\mathtt{s}_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \mathtt{h}_\beta$$

Example:

$$\mathtt{s}_{221} = \mathtt{h}_{221} - \mathtt{h}_{23} - \mathtt{h}_{41} + \mathtt{h}_5$$





$\overleftarrow{\alpha} = (2, 1, 2, 1, 1, 3, 1)$
$D(\overleftarrow{\alpha}) = \{2, 3, 5, 6, 7, 10\}$
$D(\alpha^c) = \{2, 3, 7, 10\}$
$\alpha^c = (2, 1, 4, 3, 1)$
$D(\alpha') = \{1, 4, 8, 9\}$
$\alpha' = (1, 3, 4, 1, 2)$
$\alpha = (1, 3, 1, 1, 2, 1, 2)$
$D(\alpha) = \{1, 4, 5, 6, 8, 9\}$

More non-commutative symmetric functions

$$\mathsf{e}_k = \mathsf{e}_{k-1}\mathsf{h}_1 - \mathsf{e}_{k-2}\mathsf{h}_2 + \cdots + (-1)^k \mathsf{h}_k$$
$$= \sum_{\alpha \models k} (-1)^{k+\ell(\alpha)} \mathsf{h}_\alpha$$

$$\mathsf{e}_\alpha = \mathsf{e}_{\alpha_1}\mathsf{e}_{\alpha_2} \cdots \mathsf{e}_{\alpha_{\ell(\alpha)}}$$

$$\mathsf{s}_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \mathsf{h}_\beta$$

$\{\mathsf{s}_\alpha\}_{\alpha \models n}$ $\{\mathsf{h}_\alpha\}_{\alpha \models n}$ $\{\mathsf{e}_\alpha\}_{\alpha \models n}$

are bases for NCSF of degree n

$\omega'(\mathsf{s}_\alpha) = \mathsf{s}_{\alpha'}$	$\omega^c(\mathsf{s}_\alpha) = \mathsf{s}_{\alpha^c}$	$\overleftarrow{\omega}(\mathsf{s}_\alpha) = \mathsf{s}_{\overleftarrow{\alpha}}$
$\omega'(\mathsf{h}_\alpha) = \mathsf{e}_{\overleftarrow{\alpha}}$	$\omega^c(\mathsf{h}_\alpha) = \mathsf{e}_\alpha$	$\overleftarrow{\omega}(\mathsf{h}_\alpha) = \mathsf{h}_{\overleftarrow{\alpha}}$
$\omega'(\mathsf{e}_\alpha) = \mathsf{h}_{\overleftarrow{\alpha}}$	$\omega^c(\mathsf{e}_\alpha) = \mathsf{h}_\alpha$	$\overleftarrow{\omega}(\mathsf{e}_\alpha) = \mathsf{e}_{\overleftarrow{\alpha}}$

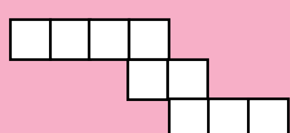
For the map $\chi : NSym \to \Lambda$ by

$$\chi(\mathsf{h}_\alpha) = h_{\alpha_1} \cdots h_{\alpha_{\ell(\alpha)}}$$
$$\chi(\mathsf{e}_\alpha) = e_{\alpha_1} \cdots e_{\alpha_{\ell(\alpha)}}$$
$$\chi(\mathsf{s}_{(1^a,b)}) = s_{(b,1^a)}$$

in general

$\chi(\mathsf{s}_\alpha) = s_\alpha$ as a skew Schur function

Example:



α=(4,2,3)

$$\chi(\mathsf{s}_\alpha) = s_{(7,5,4)/(4,3)}$$

$$s_{\lambda/\mu} = det|h_{\lambda_i - \mu_j + i - j}|$$

Hall-Littlewood symmetric functions

$$\mathbf{S}_m(s_\lambda) = s_{(m,\lambda)}$$
$$\mathbf{S}_{\lambda_1}\mathbf{S}_{\lambda_2}\cdots\mathbf{S}_{\lambda_{\ell(\lambda)}}1 = s_\lambda$$

For $V \in Hom(\Lambda, \Lambda)$

$\overline{V} = \mu \circ id \otimes (V \circ S) \circ \Delta$

$R^q(f) = q^{deg(f)} f$

for $f \in \Lambda$ of homogeneous degree

$\widetilde{V}^q = \overline{\overline{V}\ \overline{R^q}}$

$$\mathbf{H}_m = \widetilde{\mathbf{S}_m}^q$$
$$H_\lambda^q = \mathbf{H}_{\lambda_1}\mathbf{H}_{\lambda_2}\cdots\mathbf{H}_{\lambda_{\ell(\lambda)}}1$$

Theorem (Jing)

$$H_\mu^q = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$$

Non-commutative
Hall-Littlewood symmetric functions

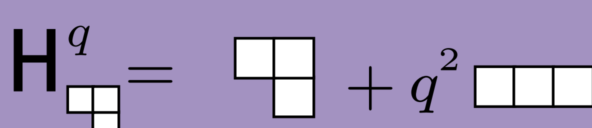
$$\mathbb{S}_m(\mathsf{s}_\alpha) = \mathsf{s}_{(\alpha,m)}$$
$$\mathbb{S}_{\alpha_{\ell(\alpha)}}\mathbb{S}_{\alpha_{\ell(\alpha)-1}}\cdots\mathbb{S}_{\alpha_1}1 = \mathsf{s}_\alpha$$

For $V \in Hom(NC\Lambda, NC\Lambda)$
$\overline{V} = \mu \circ id \otimes (V \circ S) \circ \Delta$
$R^q(f) = q^{deg(f)} f$
for $f \in NC\Lambda$ of homogeneous degree
$\widetilde{V}^q = \overline{\overline{V}\ \overline{R^q}}$

$$\mathbb{H}_m = \widetilde{\mathbb{S}_m}^q$$

$$\mathsf{H}^q_\alpha := \mathbb{H}_{\alpha_{\ell(\alpha)}}\mathbb{H}_{\alpha_{\ell(\alpha)-1}}\cdots\mathbb{H}_{\alpha_1}1$$

Example



Theorem (Z-Bergeron)

$$\mathsf{H}^q_\alpha = \sum_{\beta \geq \alpha} q^{c(\alpha,\beta^c)} \mathsf{s}_\beta$$

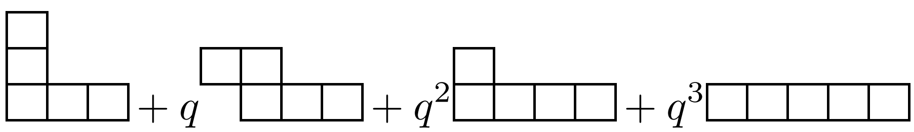
where

$$c(\alpha,\beta) = \sum_{i \in D(\alpha) \cap D(\beta)} i$$

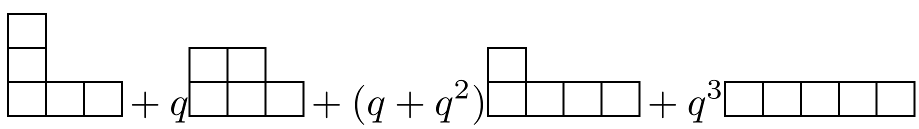
moreover

$$\chi(\mathsf{H}^q_{(1^a,b)}) = H^q_{(b,1^a)}$$

$\mathbf{H}^q_{(113)} =$



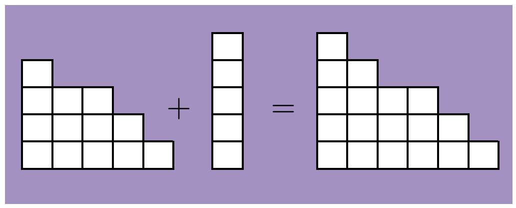
$\chi(\mathbf{H}^q_{(113)}) = H^q_{(311)} =$



$$\mathbf{H}^q_\alpha\Big|_{q=0} = \mathbf{s}_\alpha \qquad \mathbf{H}^q_\alpha\Big|_{q=1} = \mathbf{h}_\alpha$$

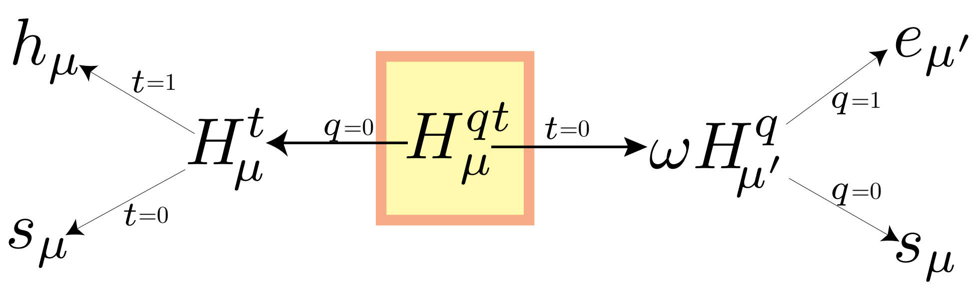
Macdonald symmetric functions

Define $\mathbf{H}^t_{1^m}(H^t_\lambda) = H^t_{\lambda+1^m}$



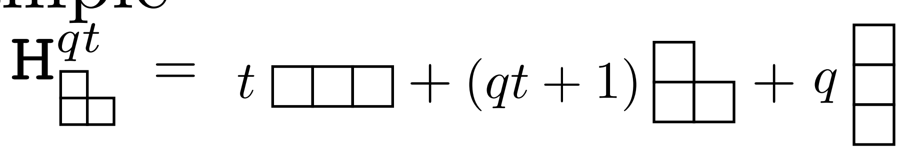
set $\mathbf{H}^{qt}_{1^m} = \widetilde{\mathbf{H}^t_{1^m}}^{\,q}$

$$H^{qt}_{\mu'} = \mathbf{H}^{qt}_{1^{\mu_1}} \mathbf{H}^{qt}_{1^{\mu_2}} \cdots \mathbf{H}^{qt}_{1^{\mu_{\ell(\mu)}}} 1 = \sum_\lambda K_{\lambda\mu'}(q,t) s_\lambda$$



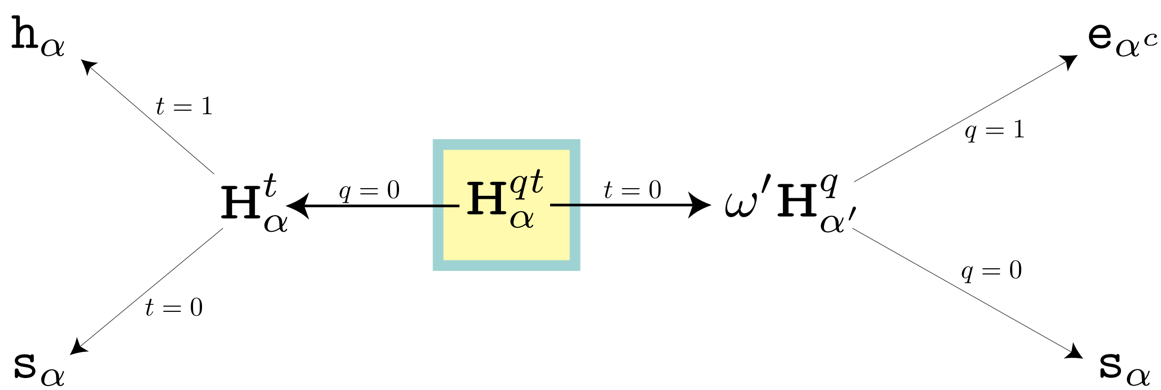
$$\omega H^{qt}_\mu = q^{n(\mu')} t^{n(\mu)} H^{\frac{1}{q}\frac{1}{t}}_\mu \qquad \omega H^{tq}_\mu = H^{qt}_{\mu'}$$

Example



A non-commutative Macdonald symmetric function

$$\mathbf{H}_{\alpha}^{qt} = \sum_{\beta} q^{c(\alpha', \overleftarrow{\beta})} t^{c(\alpha, \beta^c)} \mathbf{s}_{\beta}$$



$$\omega^c \mathbf{H}_{\alpha}^{qt} = q^{n(\alpha')} t^{n(\alpha)} \mathbf{H}_{\alpha}^{\frac{1}{q}\frac{1}{t}} \qquad \omega' \mathbf{H}_{\alpha}^{qt} = \mathbf{H}_{\alpha'}^{tq}$$

$$\chi(\mathbf{H}_{\alpha}^{qt}) = H_{(b1^a)}^{qt}$$

if $\alpha = (1^a b)$

Example