Hall-Littlewood analogs in the Q-function algebra

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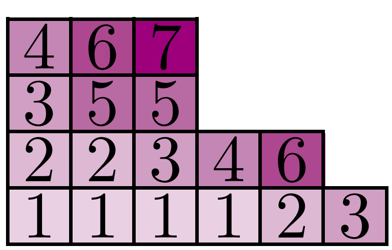
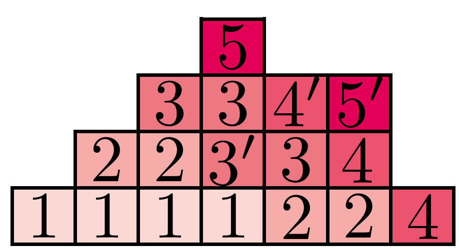




The Symmetric Functions

$$\Lambda = \mathbb{Q}[p_1, p_2, p_3, \ldots]$$
$$= \mathbb{Q}[h_1, h_2, h_3, \ldots]$$
$$deg(h_k) = k$$

The space of symmetric functions is generated algebraically by the simple homogeneous and power symmetric functions.

The Schur Functions

$$s_\lambda = det\left|h_{\lambda_i+i-j}\right|$$

Example:

$$s_{(2,2,1)} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = {h_2}^2 h_1 - h_2 h_3 - h_3 {h_1}^2 + h_4\, h_1$$

Bernstein Schur row adding operator

$$\mathbf{S}_m(s_\mu) = s_{(m,\mu)}$$

Jing's Hall-Littlewood row adding operator

$$\mathbf{H}_m(H^q_\mu) = H^q_{(m,\mu)}$$

Hopf q-ification of linear operators

Δ coproduct $\quad \Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$

μ multiplication $\quad \mu(f \otimes g) = fg$

S antipode $\quad S(p_k) = -p_k$

$$R^q(f) = q^{deg(f)} f$$

Define for $V \in Hom(\Lambda, \Lambda)$

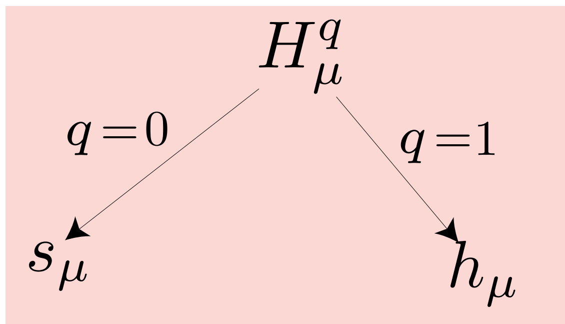
$$\overline{V} := \mu \circ id \otimes (VS) \circ \Delta$$

$$\widetilde{V}^q := \overline{\overline{V}\ \overline{R^q}}$$

$$\mathbf{H}_m = \widetilde{\mathbf{S}_m}^q$$

Hall-Littlewood symmetric functions
Kostka-Foulkes coefficients

$$H_\mu^q = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$$



$$K_{\lambda\mu}(0) = 1 \text{ if } \lambda = \mu \text{ and } 0 \text{ otherwise}$$

$$K_{\lambda\mu}(1) = h_\mu \Big|_{s_\lambda}$$

Example:

$$H^q_{(2,2,2)} = q^6 s_6 + q^4 (q+1) s_{5,1} + q^2 \left(q^2 + q + 1\right) s_{4,2} + q^3 s_{4,1,1} + q^3 s_{3,3} + q(q+1) s_{3,2,1} + s_{2,2,2}$$

Column strict tableaux (CST)

Young diagram of a partition with entries that increase weakly in the rows and strictly in the columns

Example:

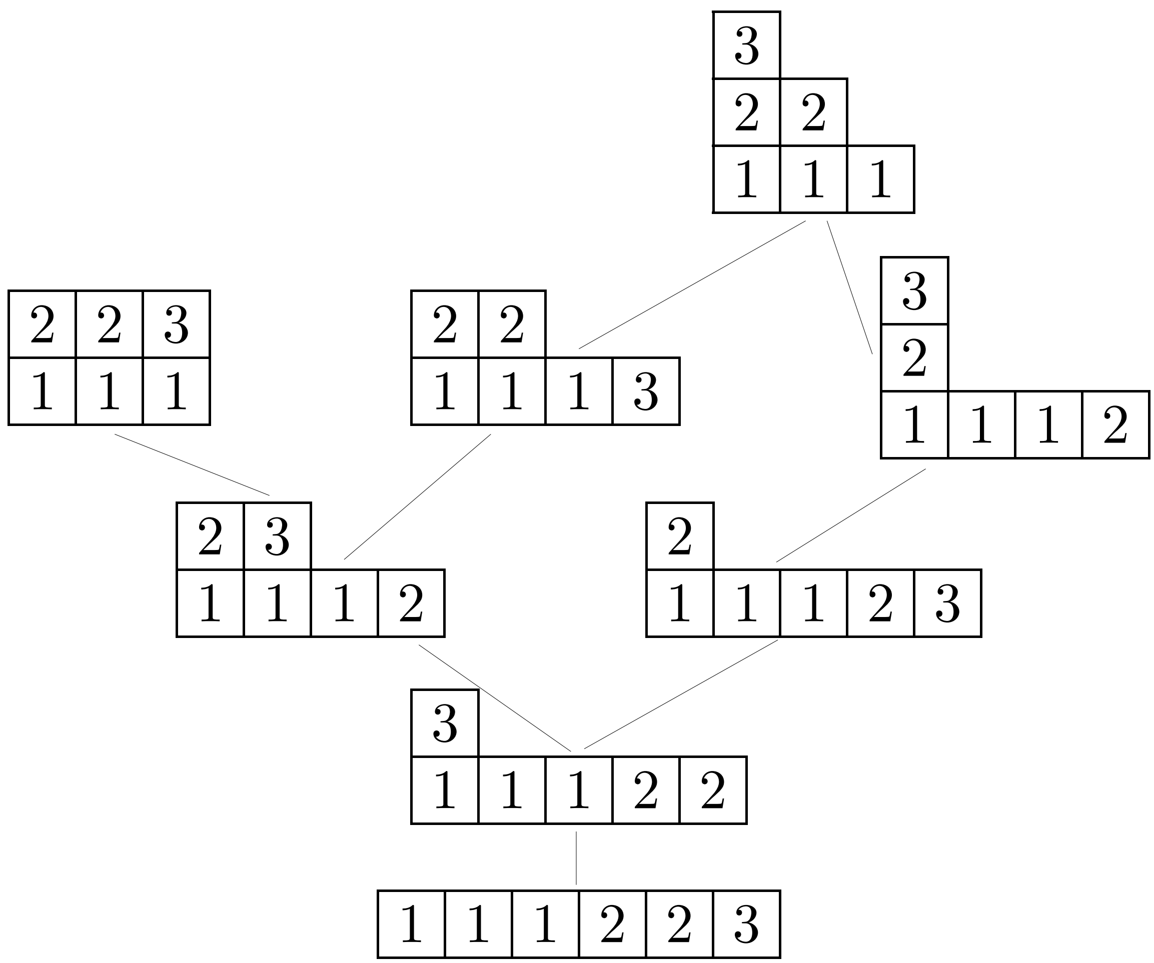
4	6	7			
3	5	5			
2	2	3	4	6	
1	1	1	1	2	3

$\lambda(T)$ = the shape of a tableau T
$\mu(T)$ = the content of a tableau T

$$h_\mu = \sum_{\substack{T \in CST \\ \mu(T) = \mu}} s_{\lambda(T)}$$

$$s_\lambda = \sum_{\substack{T \in CST \\ \lambda(T) = \lambda}} m_{\mu(T)}$$

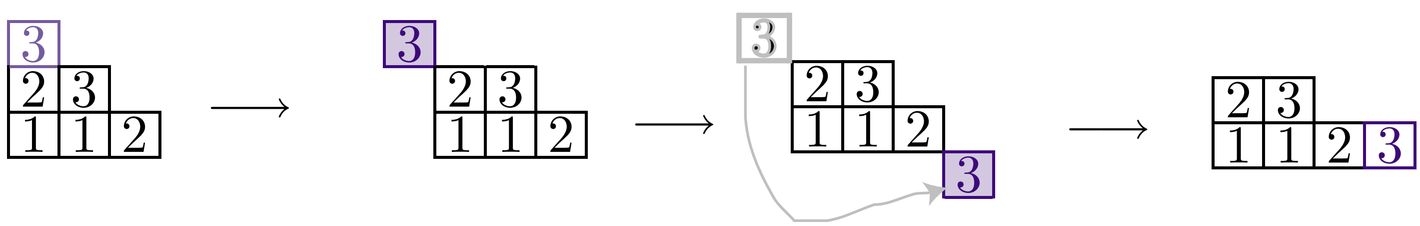
Poset structure of the column strict tableaux of content $(3,2,1)$



$$H^q_{(3,2,1)} = q^4 s_6 + q^2(q+1)\, s_{5,1} + q(q+1)\, s_{4,2} + q s_{4,1,1} + q s_{3,3} + s_{3,2,1}$$

Covering relation - cyclage

1. Pick a corner
2. column evacuate
3. row insert



If $\mu \geq \nu$, then there is an injection from CST^{μ} to CST^{ν} which preserves the statistic

This implies

$$q^{n(\mu)} H_{\nu}^{q} - q^{n(\nu)} H_{\mu}^{q}$$

is Schur positive

where $\quad n(\mu) := \sum_i (i-1)\mu_i$

Schur's Q-functions

$$\Gamma = \mathbb{Q}[p_1, p_3, p_5, \ldots] \subset \Lambda$$

$$\theta : \Lambda \longrightarrow \Gamma$$

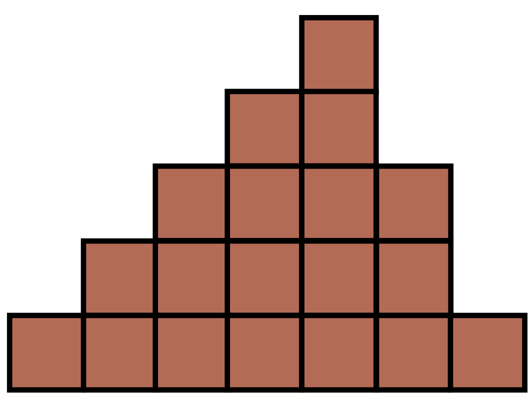
$$\theta(p_k) = (1 - (-1)^k)p_k$$

$$\theta(h_\mu) = q_\mu$$

Definition: $Q_\mu = \theta(H^q_\mu)\Big|_{q=-1}$

Basis indexed by strict partitions

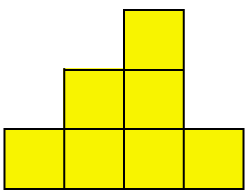
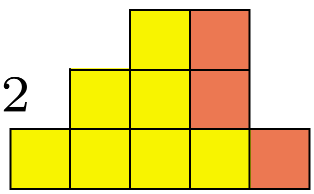
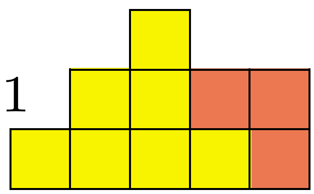
$(7, 5, 4, 2, 1)$

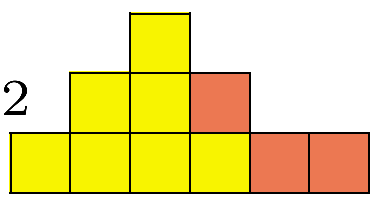
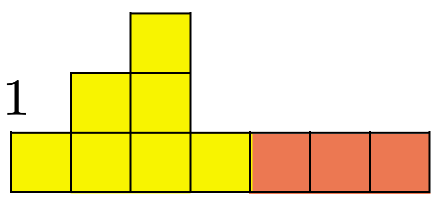


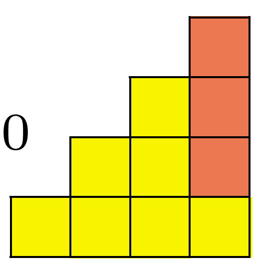
Pieri rule for the product $q_m Q_\lambda$

$$q_m Q_\mu = \sum_{\lambda/\mu \in \mathcal{H}_m} 2^{a(\lambda/\mu) - \ell(\lambda) + \ell(\mu)} Q_\lambda$$

$a(\lambda/\mu) = 1+$ the number of $1 < j \leq \ell(\lambda)$ such that $\lambda_j > \mu_j$ and $\mu_{j-1} > \lambda_j$

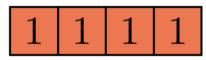
q_3  $= 2^2$  $+ 2^1$ 

$+ 2^2$  $+ 2^1$ 

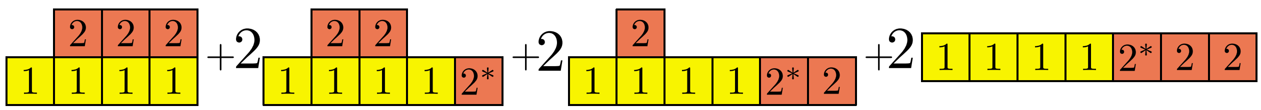
$+ 2^0$ 

$q_\lambda \longrightarrow$ Schur's Q-functions

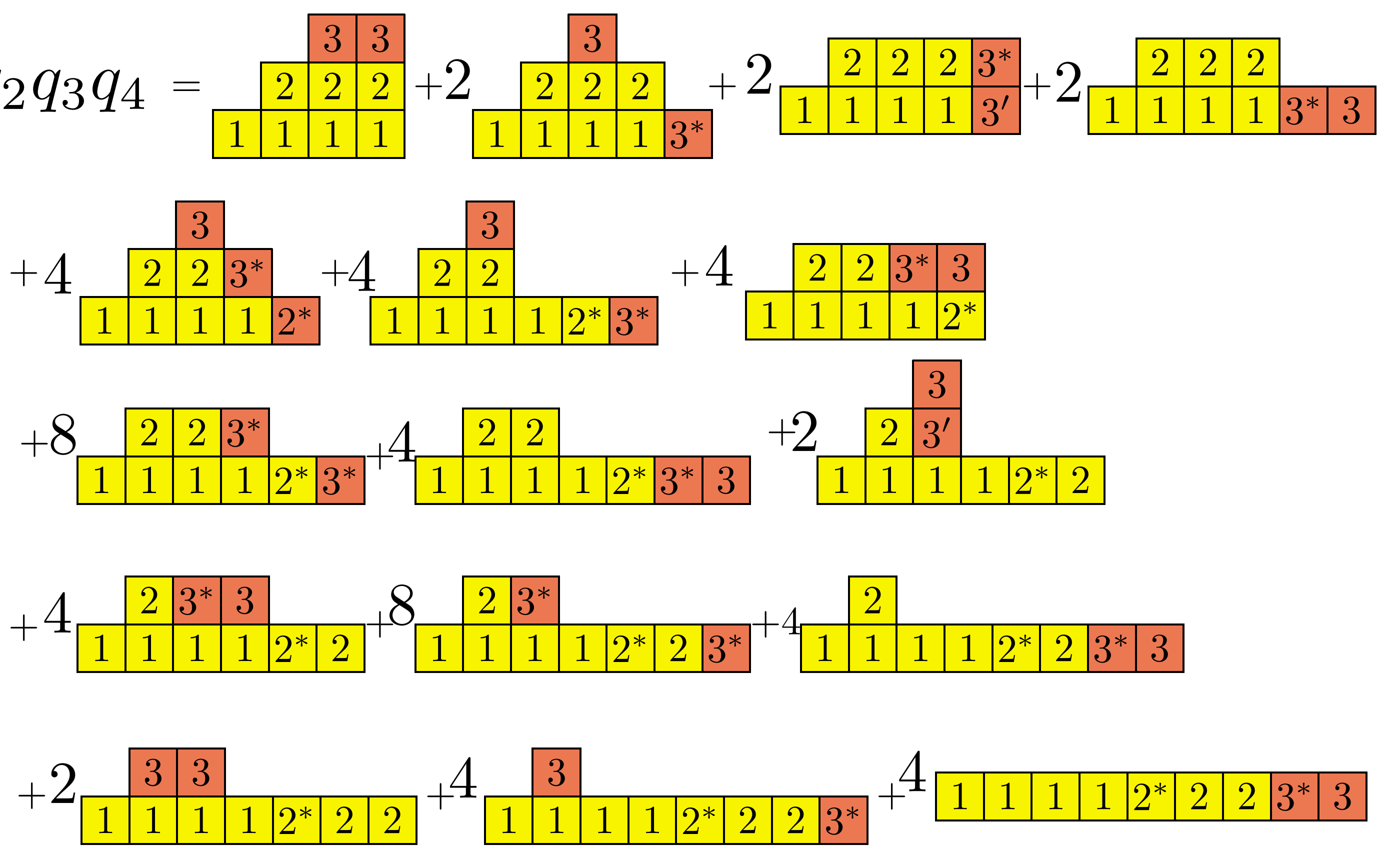
$$q_4 q_3 q_2$$

$q_4 =$ 

$$= Q_4$$

$q_3 q_4 =$ 

$$= 2\,Q_7 + 2\,Q_{6,1} + 2\,Q_{5,2} + Q_{4,3}$$

$q_2 q_3 q_4 =$ 

$$= 4\,Q_9 + 8\,Q_{8,1} + 14\,Q_{7,2} + 14\,Q_{6,3} + 6\,Q_{6,2,1} + 6\,Q_{5,4} + 6\,Q_{5,3,1} + Q_{4,3,2}$$

Marked Shifted Tableaux (MST)

Shifted Young diagram with entries weakly increasing in the rows and columns in the alphabet

$$1' < 1 < 2' < 2 < 3' < 3 < \cdots$$

with the configurations $\begin{array}{|c|}\hline k \\ \hline k \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline k' & k' \\ \hline\end{array}$ excluded

and diagonal entries are not marked (MST')

Example:

			5			
		3	3	4′	5′	
	2	2	3′	3	4	
1	1	1	1	2	2	4

$\lambda(T) =$ the shape of a tableau T
$\mu(T) =$ the content of a tableau T

$$q_\mu = \sum_{\substack{T \in MST' \\ \mu(T) = \mu}} Q_{\lambda(T)}$$

$$Q_\lambda = 2^{\ell(\lambda)} \sum_{\substack{T \in MST' \\ \lambda(T) = \lambda}} m_{\mu(T)}$$

Jing

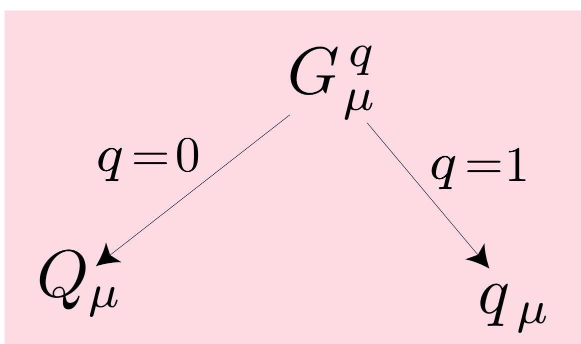
$$\mathbf{Q}_m = \theta(\mathbf{H}_m^{q=-1})$$

$$\mathbf{Q}_m(Q_\lambda) = Q_{(m,\lambda)} \qquad \text{for } m > \lambda_1$$

$$\mathbf{G}_m := \widetilde{\mathbf{Q}_m}^{q}$$

$$G_\mu^q := G_{\mu_1} G_{\mu_2} \cdots G_{\mu_k} 1$$

$$G_\mu^q = \sum_\lambda L_{\lambda\mu}(q) Q_\lambda$$



Example :

$$G^q_{(4,3,1)} = 4\,q^5 Q_8 + \left(2\,q^3 + 4\,q^4\right) Q_{7,1} + \left(4\,q^3 + 4\,q^2\right) Q_{6,2}$$
$$+ \left(4\,q^2 + 2\,q\right) Q_{5,3} + 2\,q Q_{5,2,1} + Q_{4,3,1}$$

Some Results

Theorem (Tudose & Zabrocki)

$$L_{\alpha,(n,\mu)}(q) = \sum_{s=1}^{t:\alpha_t \geq n} (-1)^{s-1} q^{\alpha_s - n} \sum_{\lambda:\lambda/\alpha^{(s)} \in \mathcal{H}_{(\alpha_s - n)}} 2^{a(\lambda/\alpha^{(s)})} L_{\lambda\mu}(q)$$

where $n > \mu_1$ and $\alpha^{(s)}$ is α with part α_s removed.

Proposition

$$deg_q L_{\lambda\mu}(q) = q^{n(\mu)-n(\lambda)}$$

Proposition

$$2^{\ell(\mu)-\ell(\lambda)} \text{ divides } L_{\lambda\mu}(q)$$

Conjectures

Conjecture

$$G^q_\mu = \sum_{T \in MST^\mu} q^{d(T)} Q_{\lambda(T)}$$

In particular,
$L_{\lambda\mu}(q)$ is a polynomial in q with non-negative integer coefficients

Conjecture
If $\mu > \lambda$ are strict partitions then

$$2^{\ell(\mu)} q^{n(\mu)} G^q_\lambda - 2^{\ell(\lambda)} q^{n(\lambda)} G^q_\mu$$

is Q-Schur positive