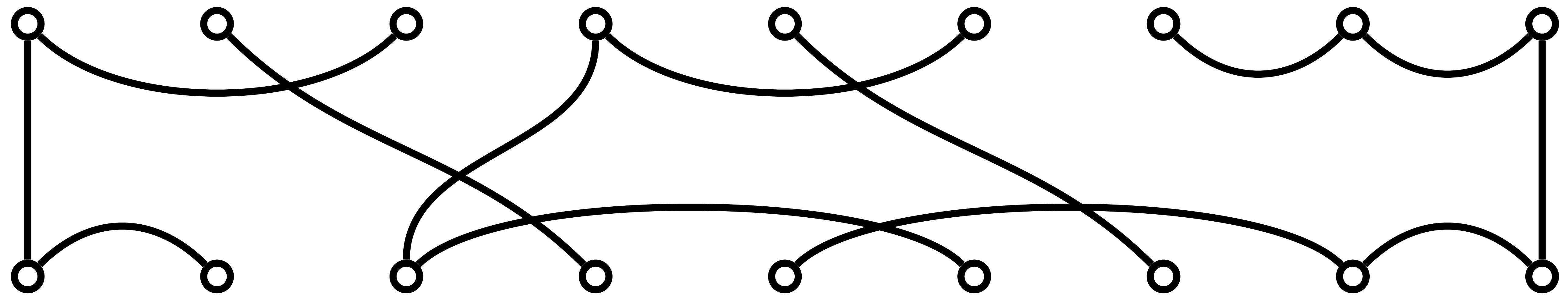
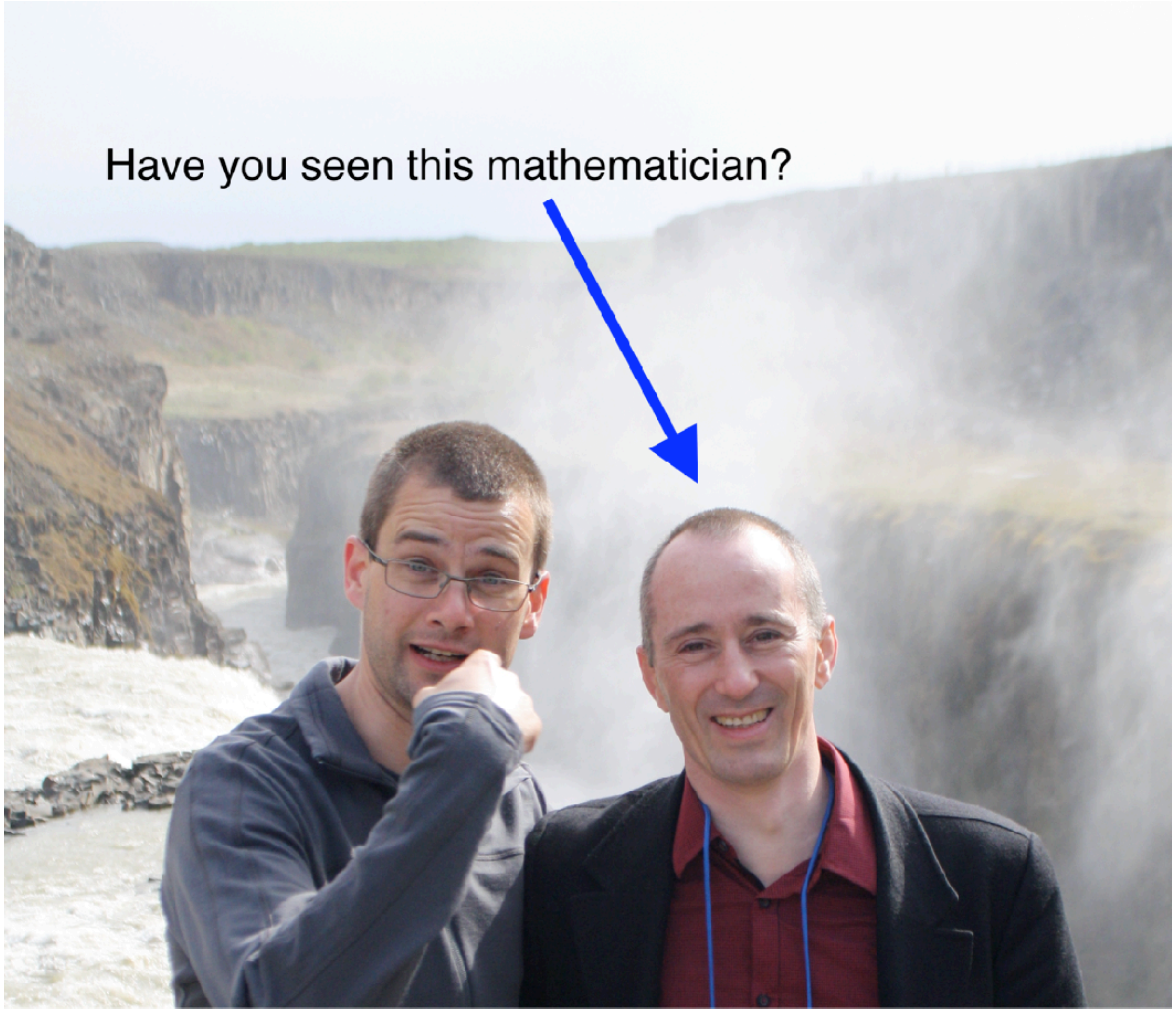
Representation theory of the monoid of uniform block permutations

Mike Zabrocki
joint work with Rosa Orellana, Franco Saliola and Anne Schilling





Happy birthday Ole!

Restriction from general linear group to group of permutation matrices

Motivation: $$\mathrm{Res}^{GL_3}_{\mathfrak{S}_3} W^{(4,4,1)}_{GL_3} \cong W^{(1,1,1)}_{\mathfrak{S}_3} \oplus (W^{(2,1)}_{\mathfrak{S}_3})^{\oplus 3} \oplus (W^{(3)}_{\mathfrak{S}_3})^{\oplus 3}$$

$$\begin{array}{cccc} 3 & & & \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{cccc} 3 & & & \\ 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{cccc} 3 & & & \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{cccc} 3 & & & \\ 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{cccc} 3 & & & \\ 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{array}$$

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$$\mathfrak{S}_3 \subset GL_3(\mathbb{C})$$

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$$\begin{array}{l} 3 \\ 2 \\ 1 \end{array} \quad \begin{array}{ll} 3 \\ 1&2 \end{array} \quad \begin{array}{ll} 2 \\ 1&3 \end{array} \quad \begin{array}{lll} 1&2&3 \end{array}$$

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$$\mathrm{Res}^{GL_n}_{\mathfrak{S}_n} W^{\lambda}_{GL_n} \cong \bigoplus_{\mu \vdash n} (W^{\mu}_{\mathfrak{S}_n})^{\oplus r_{\lambda\mu}}$$

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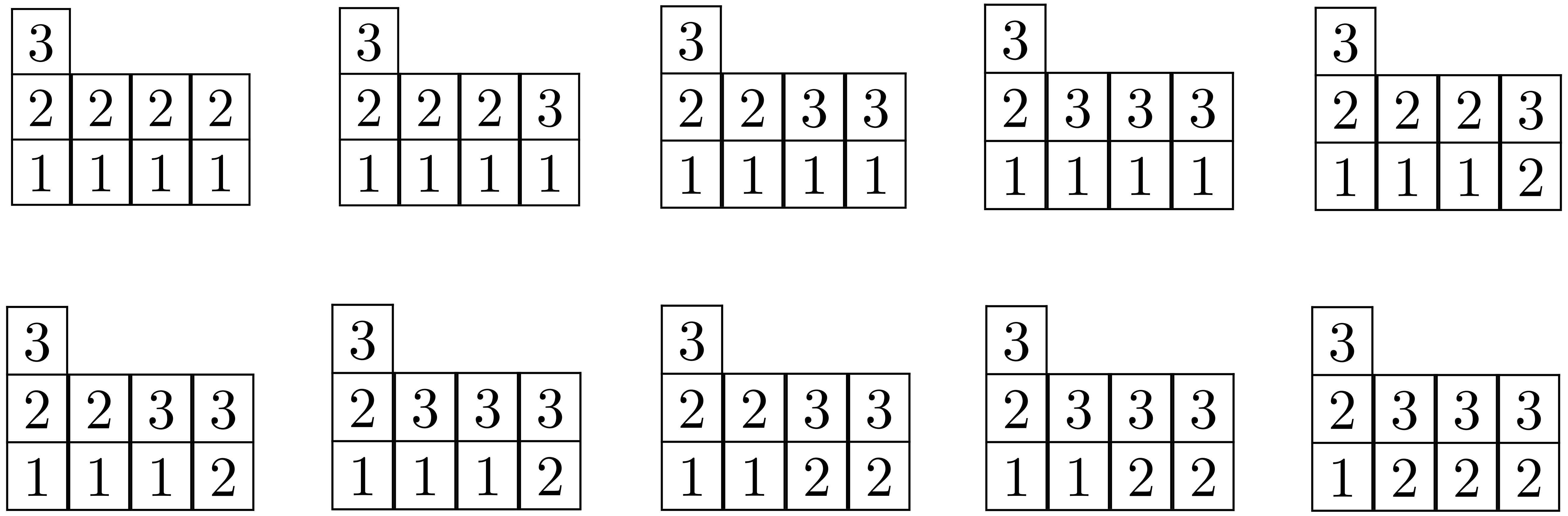

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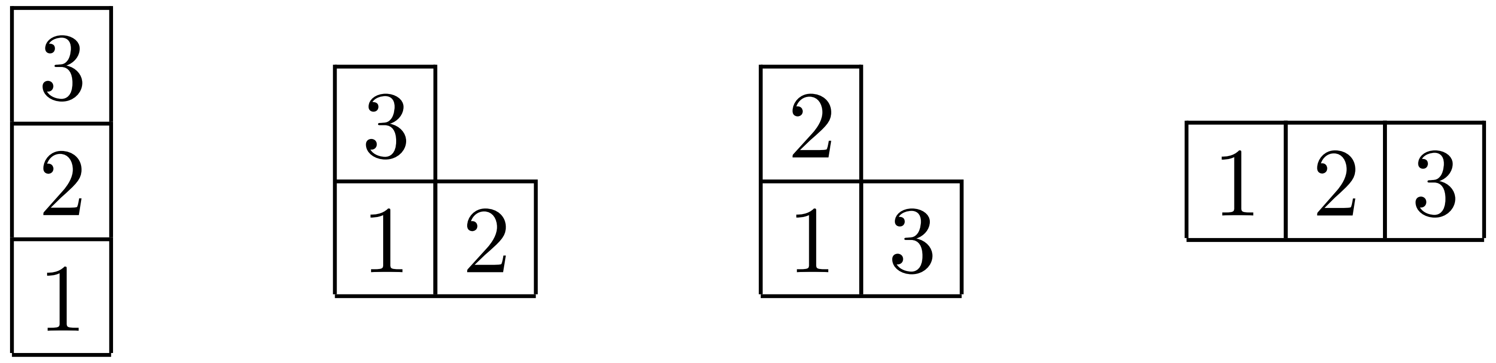

Littlewood (1950's): $r_{\lambda\mu} = \langle s_\lambda, s_\mu[1 + s_1 + s_2 + s_3 + \ldots] \rangle$

plethysm is the answer you are looking for
warning: very old problem and hard

Roads were made for journeys, not destinations.

-Confucius

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Paraphrase:

It may be a while before we figure out a satisfactory solution to this problem,

but at least the mathematics along the way has been interesting.

From classical groups to diagram algebras

$$GL_n(\mathbb{C})$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

matrices

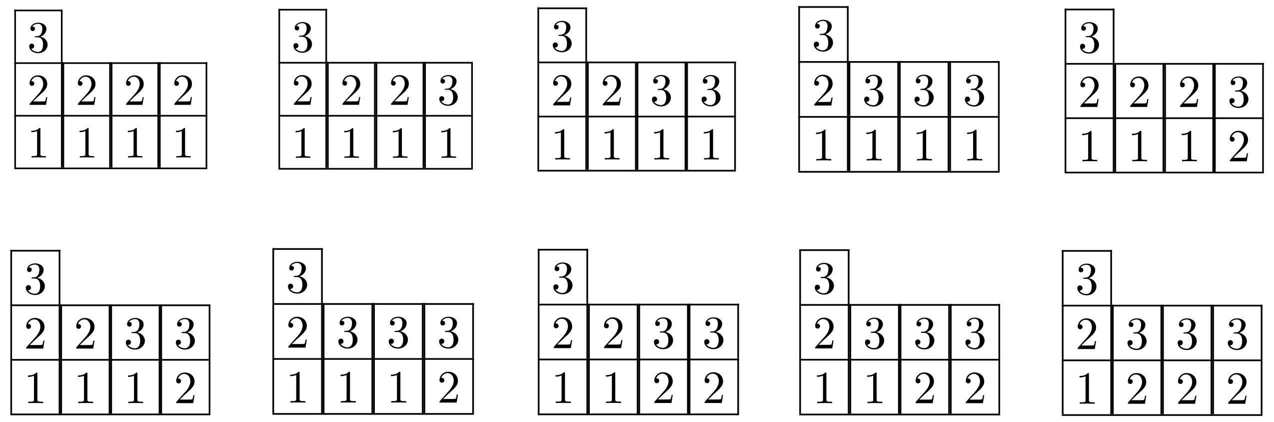
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

permutation matrices $\mathfrak{S}_n$

$$\mathfrak{S}_n \hookrightarrow GL_n(\mathbb{C})$$

From classical groups to diagram algebras

semistandard Young tableaux



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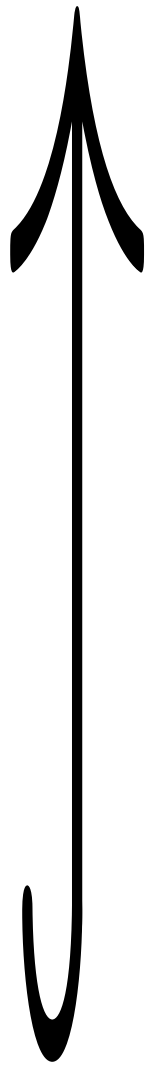

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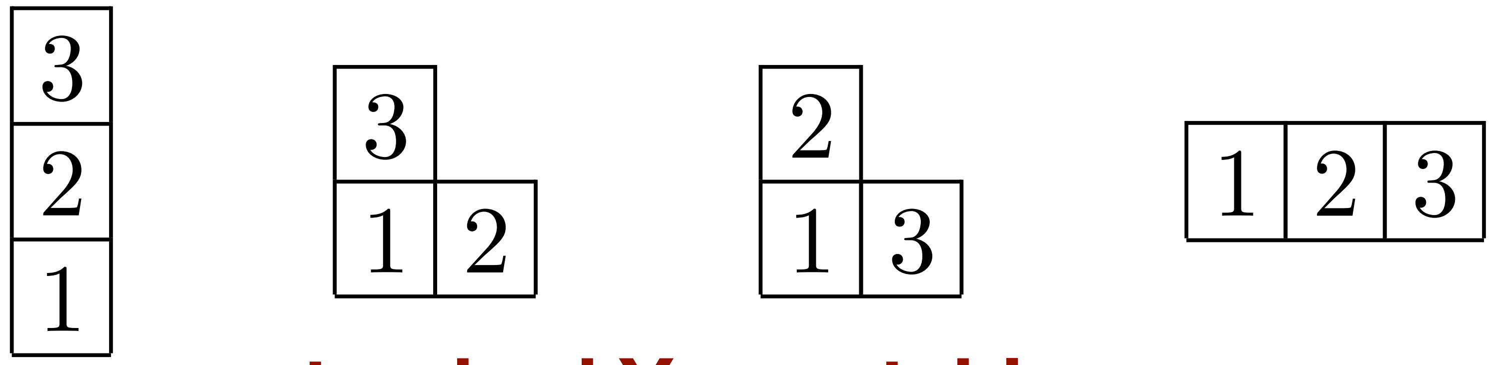
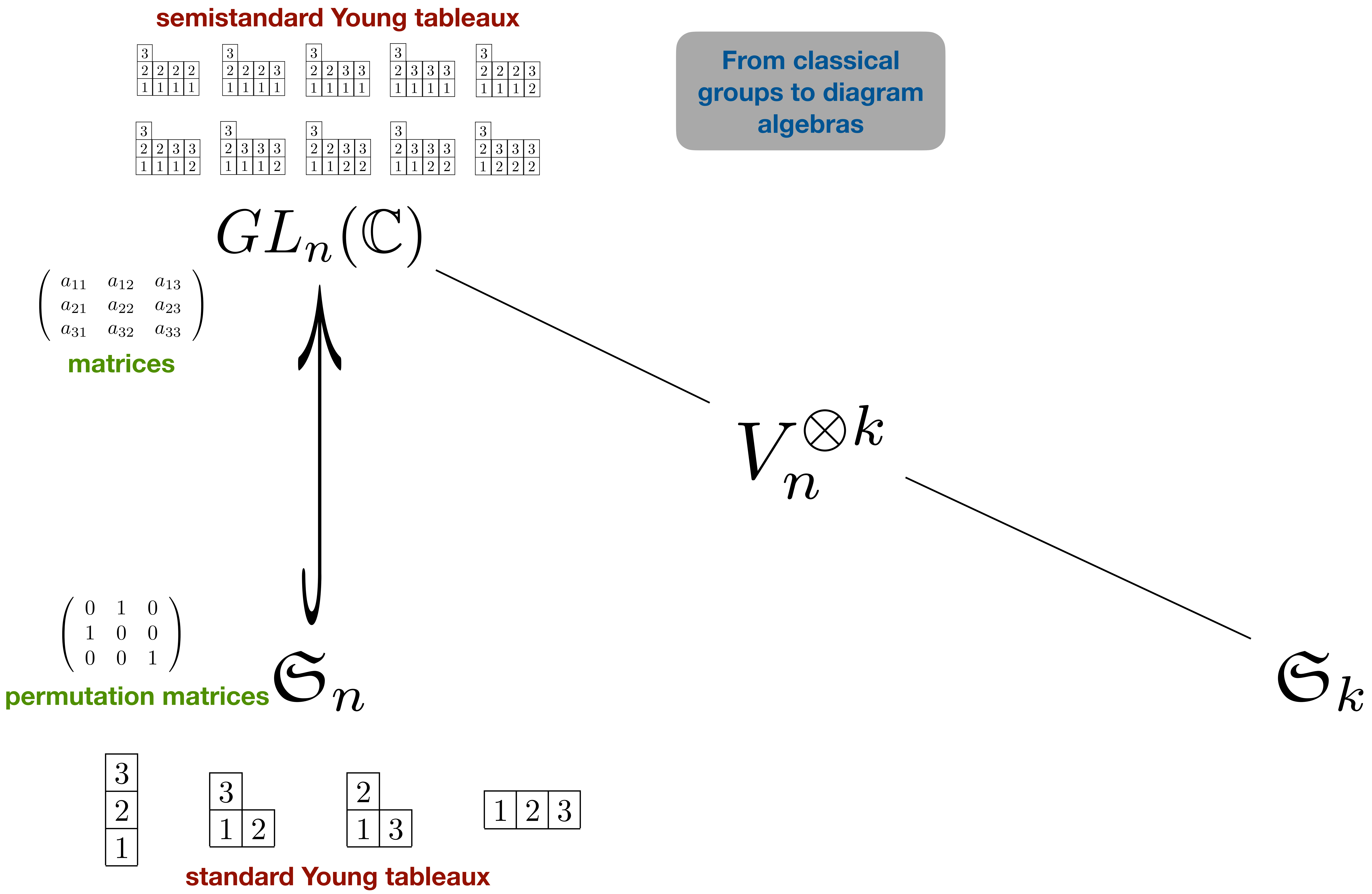


standard Young tableaux


From classical groups to diagram algebras
semistandard Young tableaux
$GL_n(\mathbb{C})$
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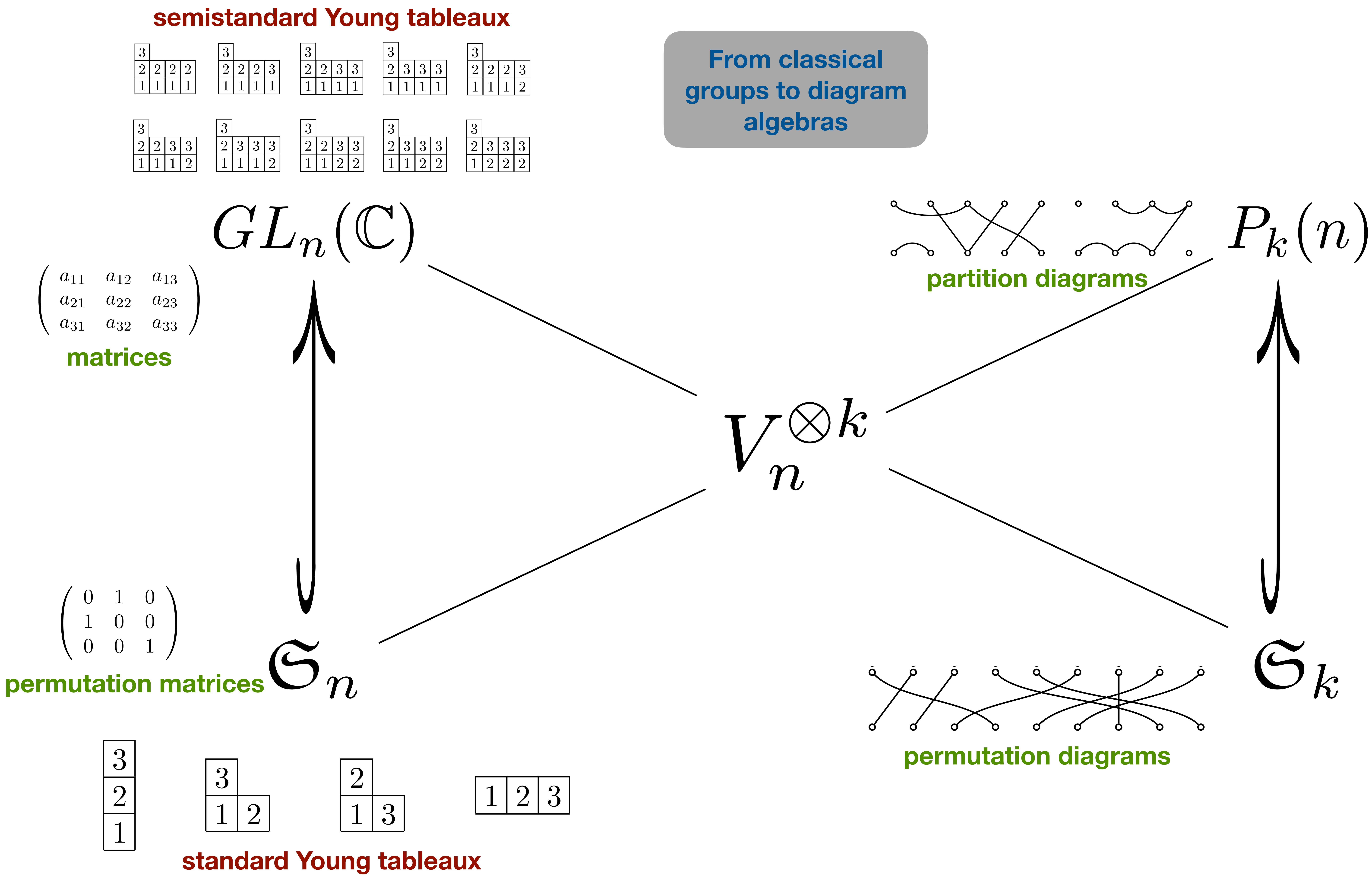

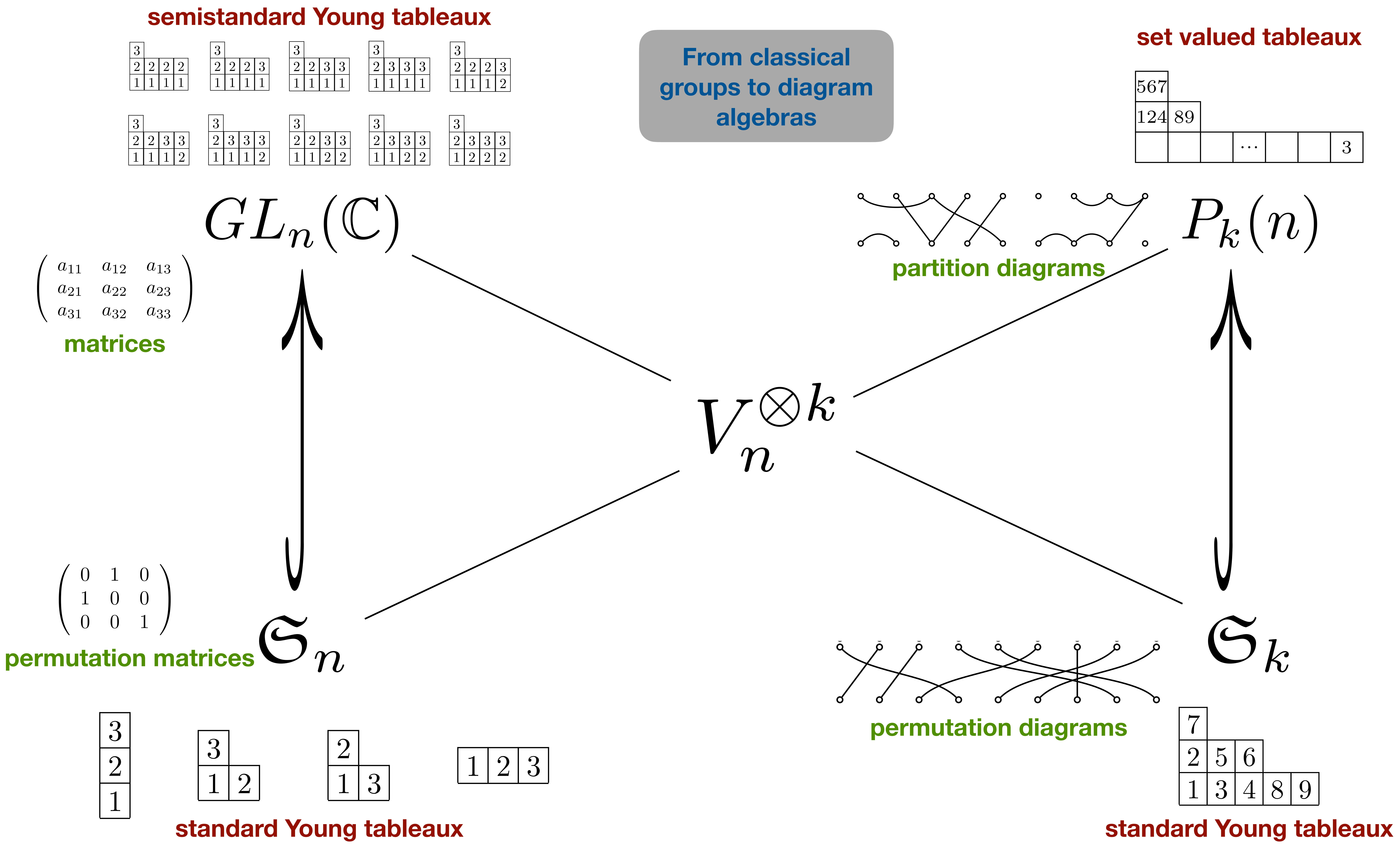
From classical groups to diagram algebras
semistandard Young tableaux
set valued tableaux
$GL_n(\mathbb{C})$
$P_k(n)$
partition diagrams
matrices
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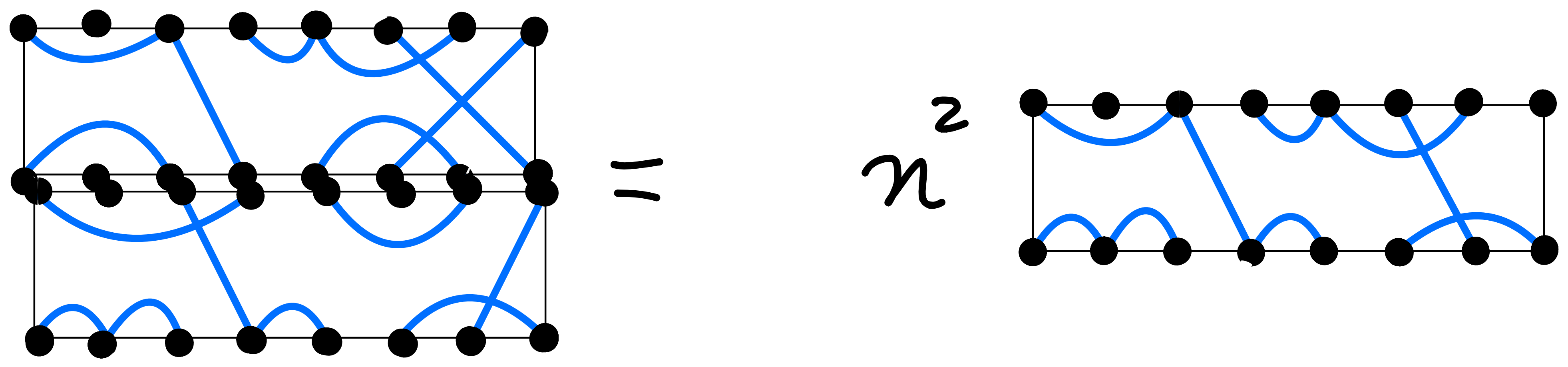
Summary of partition algebra

The partition algebra was introduced in the early 90's independently by Martin and Jones



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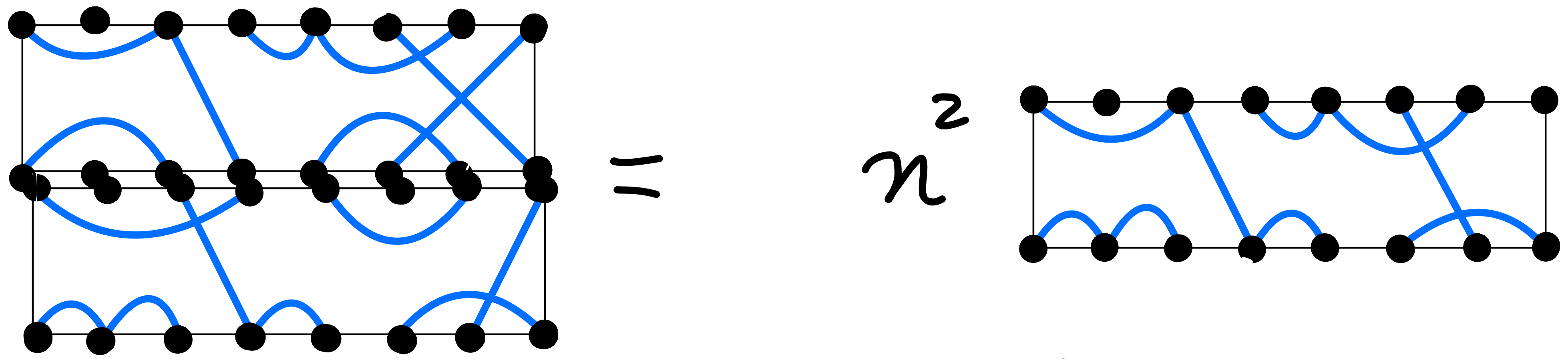

A difficulty with understanding the representation theory of the general linear group in terms of the symmetric group is we pass from an infinite group to a finite one. However, the partition algebra has dimension equal to the Bell numbers and the symmetric group algebra is dimension equal to n!

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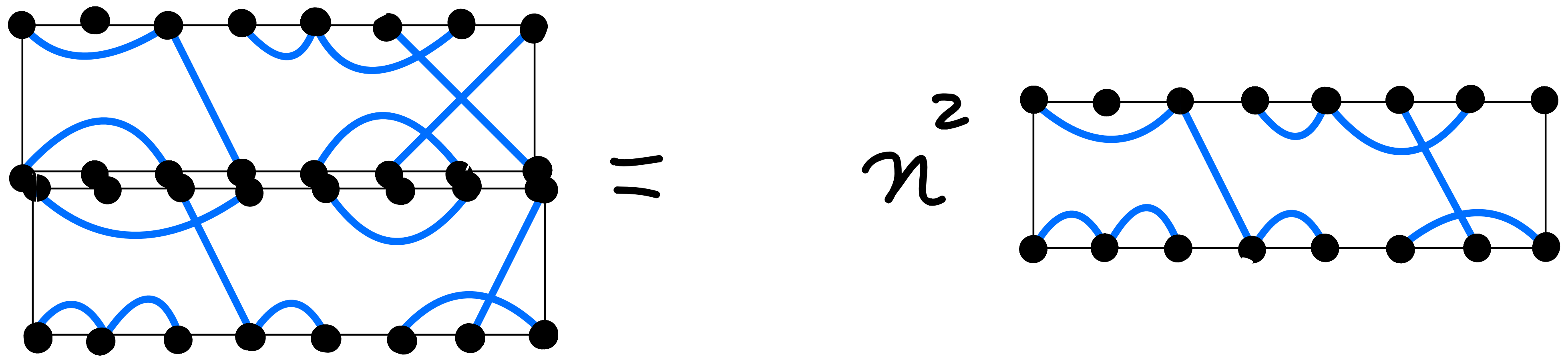

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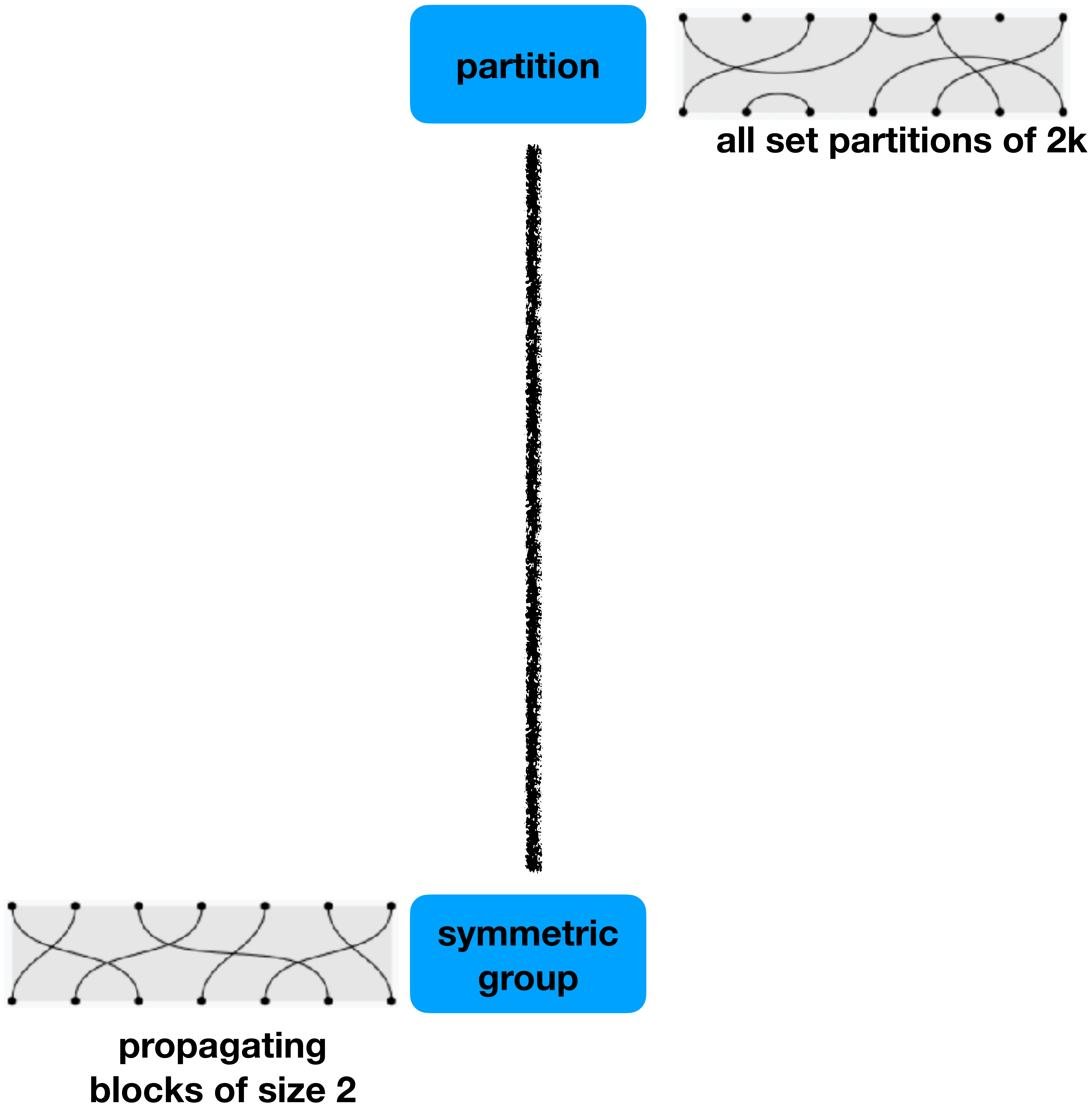
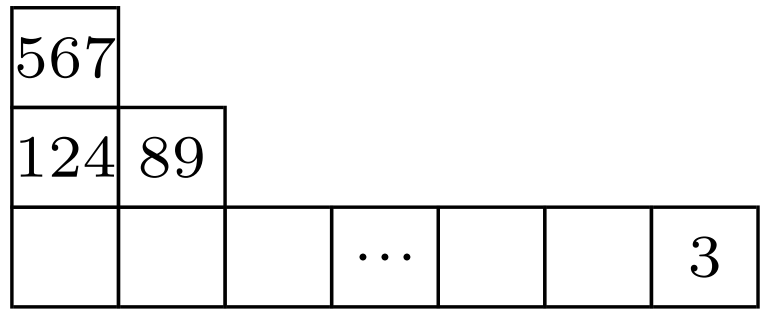
$$\mathrm{Res}^{GL_n}_{\mathfrak{S}_n} W^{\lambda}_{GL_n} \cong \bigoplus_{\mu \vdash n} (W^{\mu}_{\mathfrak{S}_n})^{r_{\lambda\mu}} \longleftarrow \text{see-saw pair} \longrightarrow \mathrm{Res}^{P_k(n)}_{\mathfrak{S}_k} W^{\mu}_{P_k(n)} \cong \bigoplus_{\lambda \vdash n} (W^{\lambda}_{\mathfrak{S}_k})^{r_{\lambda\mu}}$$


partition
all set partitions of 2k
symmetric group
propagating blocks of size 2

set valued tableaux



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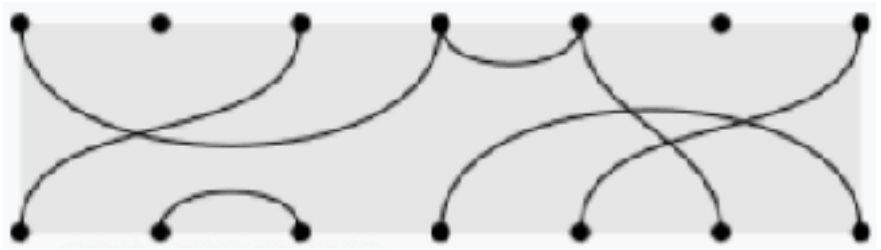

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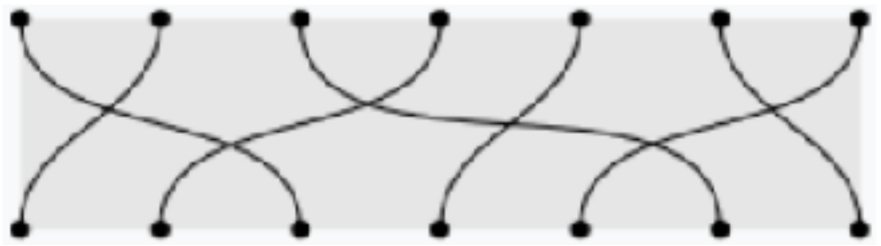

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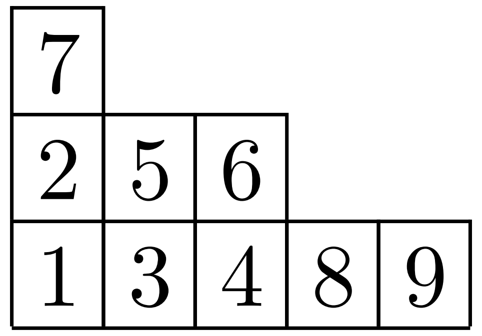
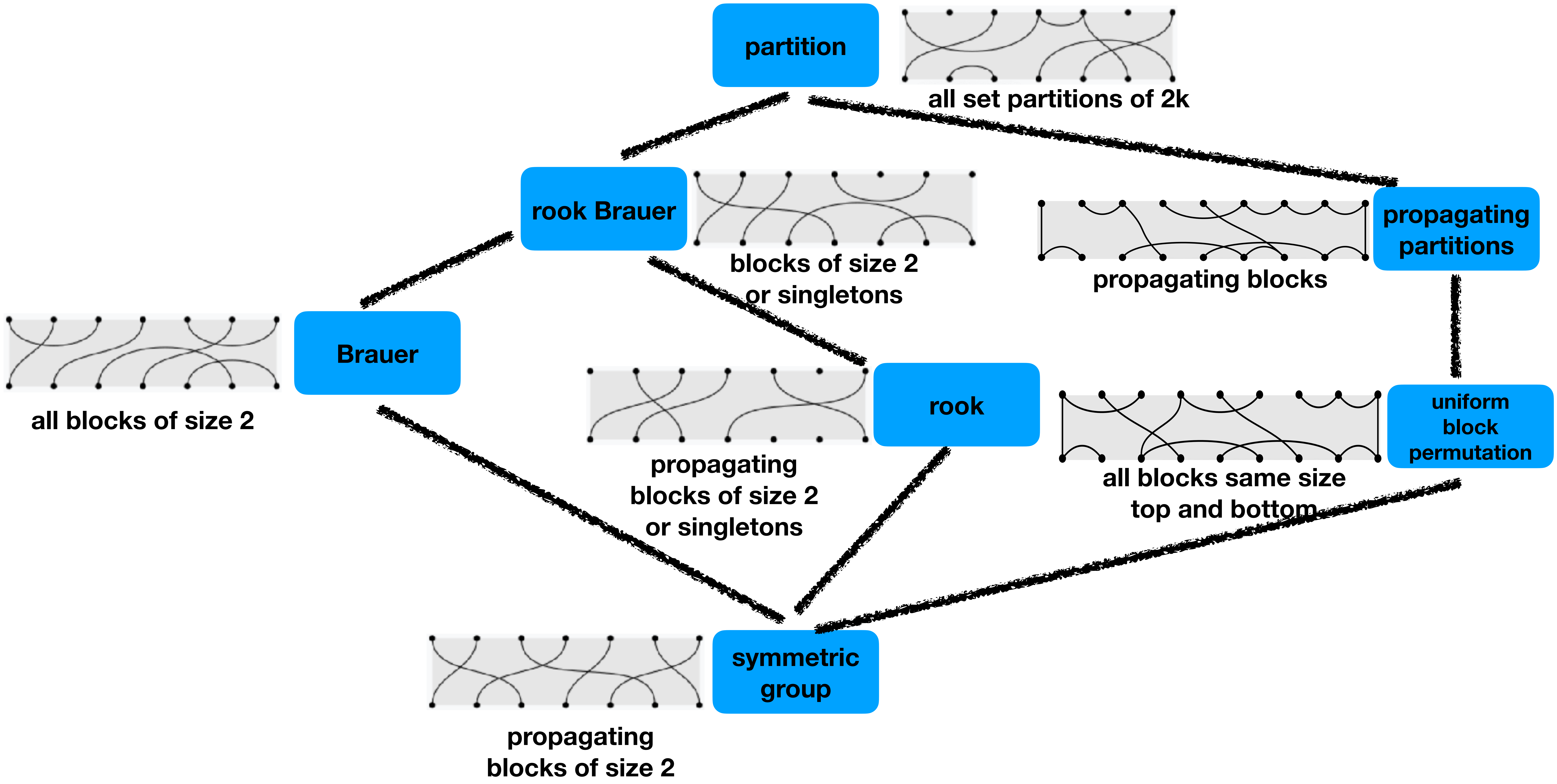


standard Young tableaux



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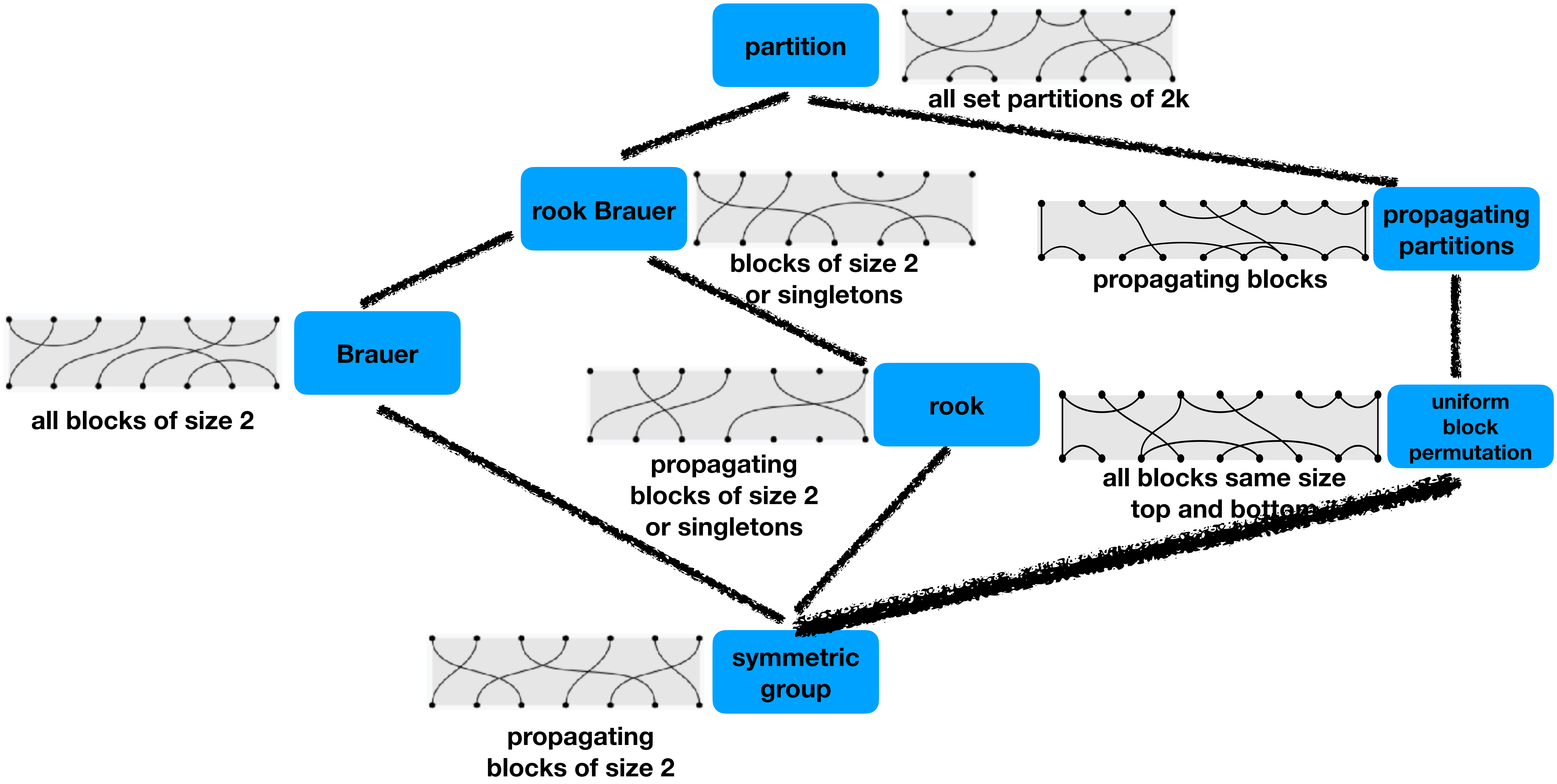
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Representation theory of uniform block partitions

$\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ each $\lambda^{(i)}$ is a partition

$|\lambda^{(1)}| + 2|\lambda^{(2)}| + \cdots + k|\lambda^{(k)}| = k$

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Example:

typical basis element

$\vec{\lambda} = ((2,1),(2,2),(1,1))$

$$\mathbf{T} = \left(\begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\, , \quad \begin{array}{|c|c|} \hline 67 & ab \\ \hline 45 & 89 \\ \hline \end{array}\, , \quad \begin{array}{|c|} \hline fgh \\ \hline cde \\ \hline \end{array} \right)$$

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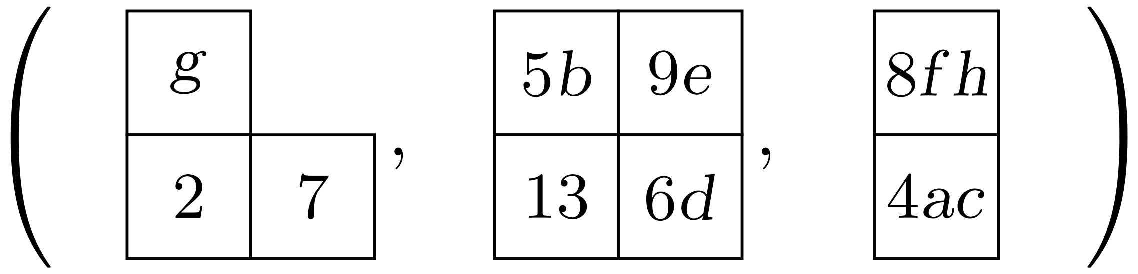
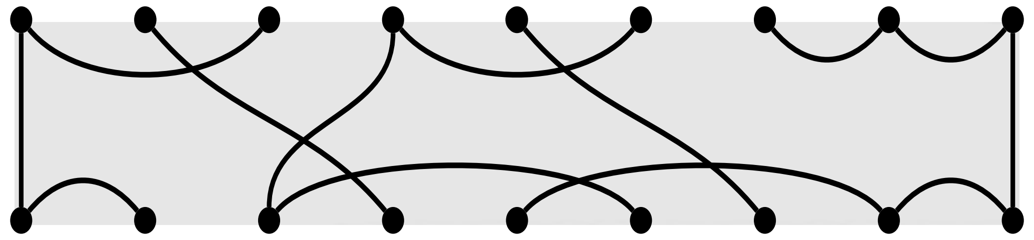
Theorem (Orellana-Saliola-Schilling-Z ’22)

$$\mathrm{Res}^{\mathrm{UBP_k}}_{\mathfrak{S}_k} W^{\vec{\lambda}}_{\mathrm{UBP}_k} \cong \bigoplus_{\mu \vdash k} (W^{\mu}_{\mathfrak{S}_k})^{\oplus a_{\vec{\lambda}\mu}} \qquad a_{\vec{\lambda}\mu} = \langle s_\mu, s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \cdots s_{\lambda^{(k)}}[s_k] \rangle$$

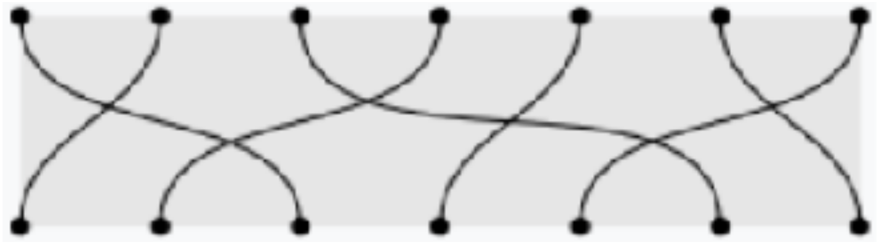
tuples of set valued tableaux



uniform block permutation



all blocks same size top and bottom



symmetric group

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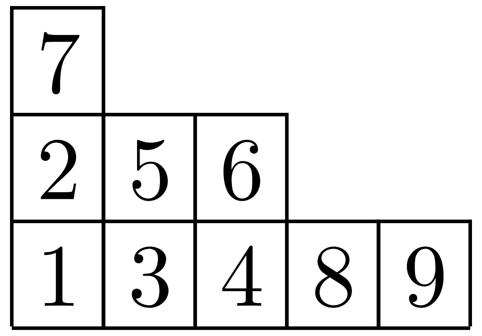

standard Young tableaux

Structure of uniform block partition monoid

- the uniform block permutation (UBP) algebra is a monoid algebra
- monoid theory $\Longrightarrow$ UBP_k is a union of $\mathcal{J}$-classes

$$x \equiv_{\mathcal{J}} y \Longleftrightarrow MxM = MyM$$

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$\mathcal{J}$-classes are indexed by partitions of k

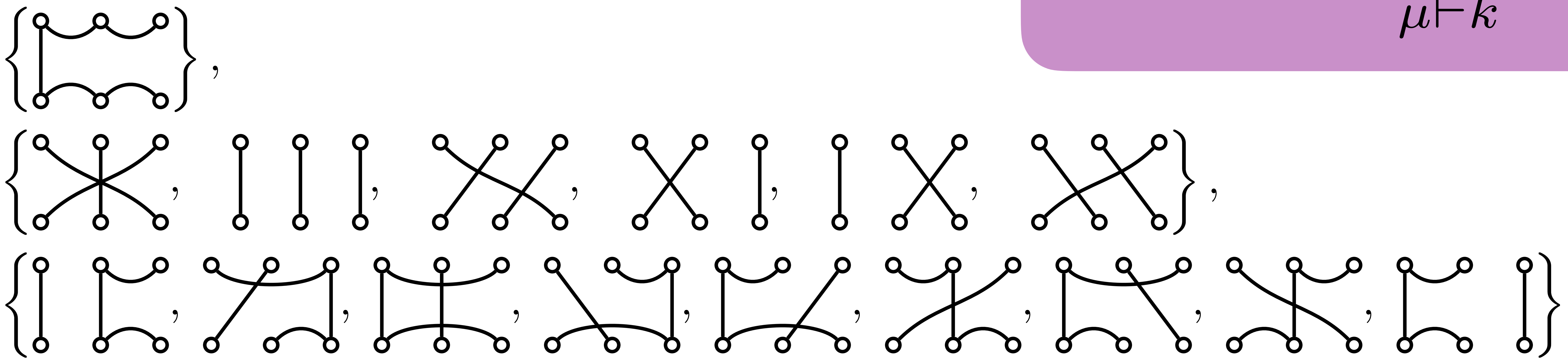
J_μ = sorted list of block sizes equal to μ

$$\mathrm{UBP}_k = \bigcup_{\mu \vdash k} J_\mu$$

$J_{(3)} =$

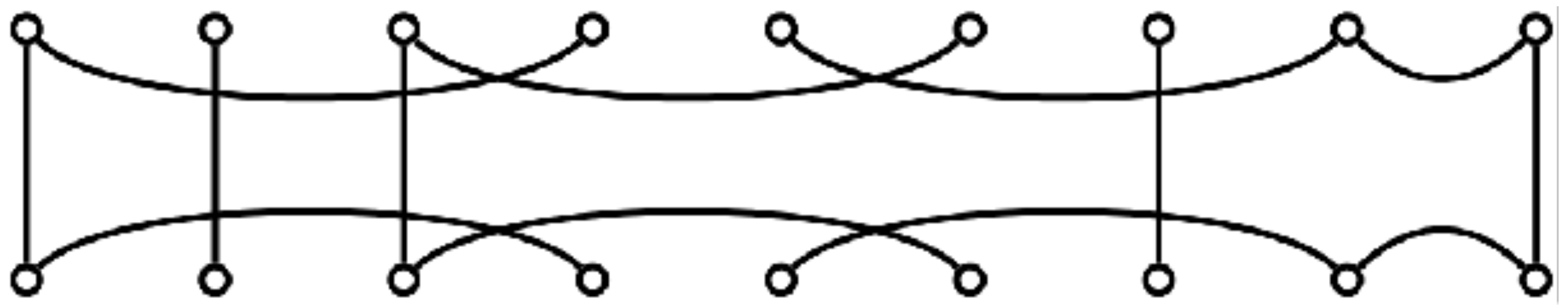
$J_{(1,1,1)} =$

$J_{(2,1)} =$



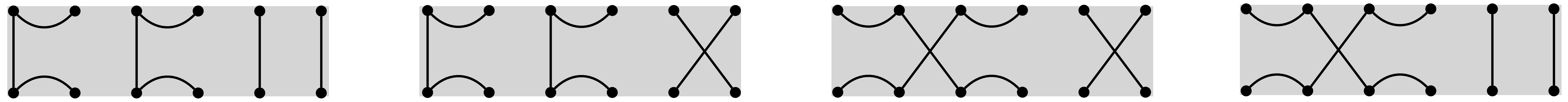
Structure of uniform block partition monoid

- the uniform block permutation (UBP) algebra is a monoid algebra
- For each idempotent e of a monoid there is a maximal subgroup G_e containing e

typical idempotent $e_{2|7|14|36|589} =$ 

representative

G_λ = maximal subgroup whose sizes of parts are (in order) the partition λ

$G_{2211} = \Big\{$  $\Big\}$

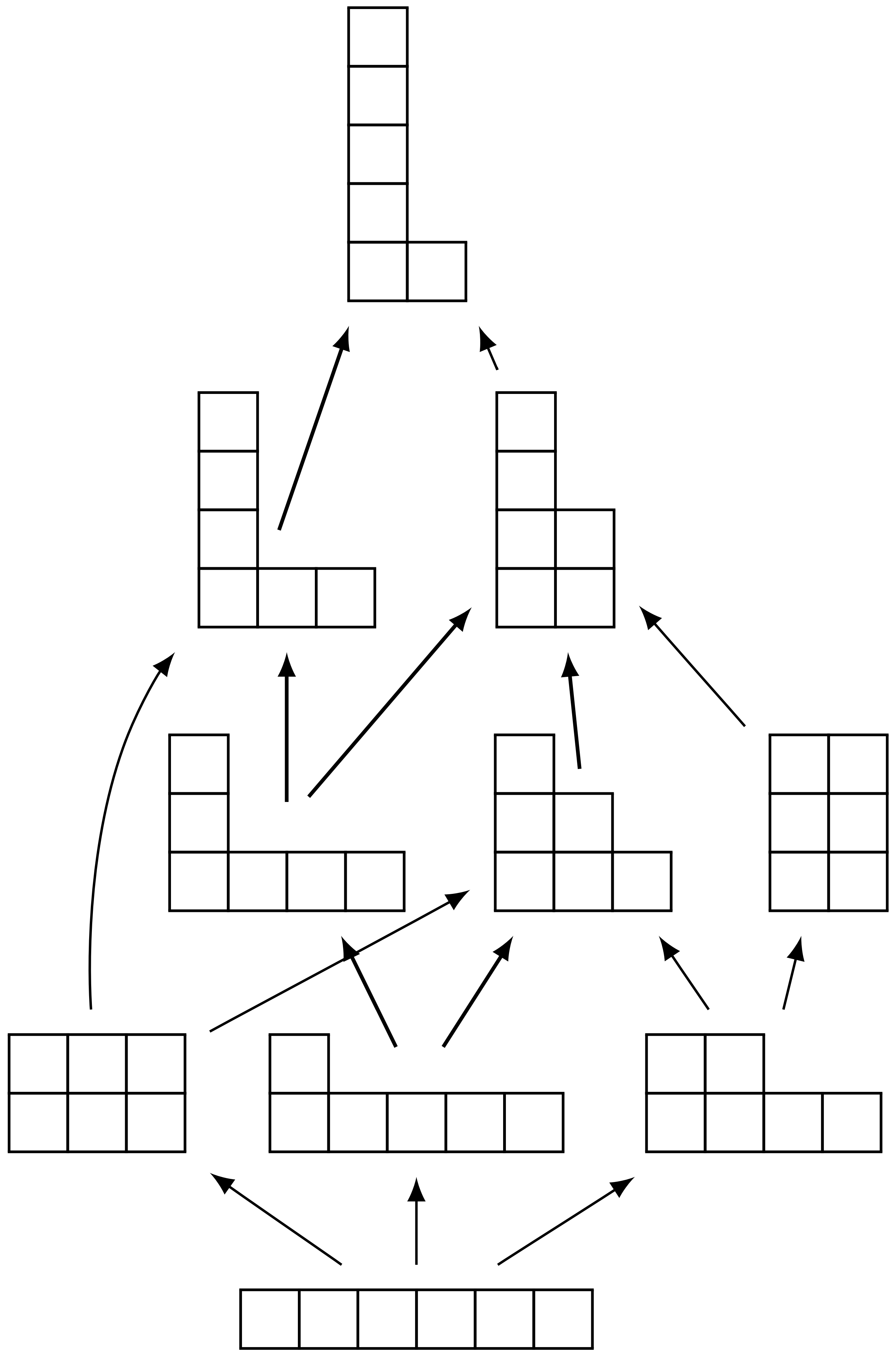
$$G_\lambda \cong \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \mathfrak{S}_{a_k}$$

$$\lambda = (k^{a_k} \cdots 2^{a_2} 1^{a_1})$$

the maximal subgroups are isomorphic to direct products of symmetric groups

For $\nu \vdash k$ with $\nu \neq 1^k$, let ς_ν be the smallest part of ν not equal to 1. Then

$$\mu \preceq \lambda \qquad \text{if and only if} \qquad \mu \text{ is coarser than } \lambda \text{ and } \varsigma_\mu \geqslant \varsigma_\lambda$$



Hasse diagram for $(\mathcal{P}_6 \setminus \{1^6\}, \preceq)$

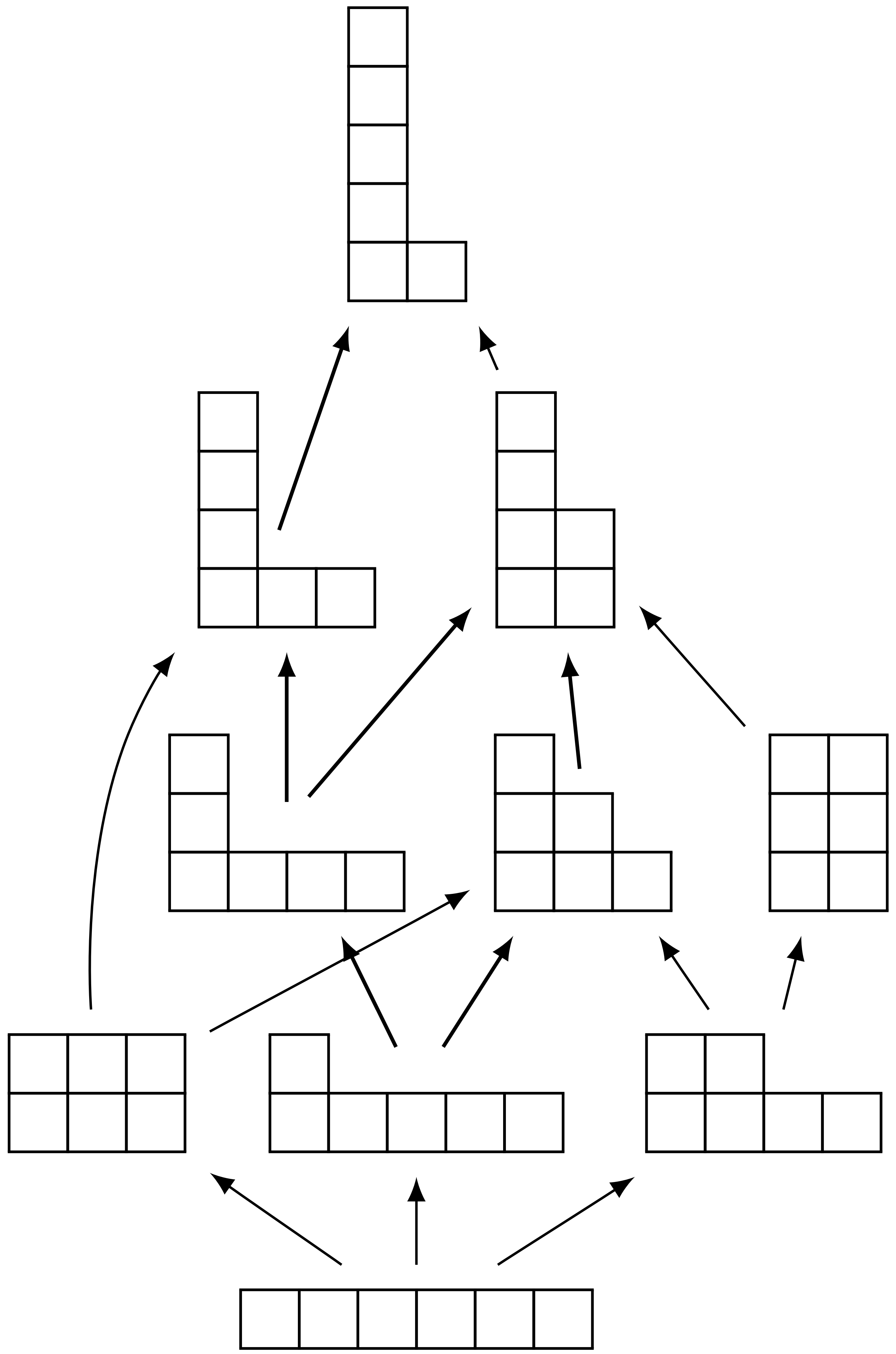
Theorems (Orellana-Saliola-Schilling-Z ’25)

For k a positive integer, the set of submonoids of UBP_k that contain $\mathfrak{S}_k$ forms a distributive lattice with the operations of union and intersection.

Every submonoid S of UBP_k that contains $\mathfrak{S}_k$ is of the form

$$S = \mathfrak{S}_k \cup \bigcup_{\mu \in I} J_\mu$$

for some down set I of the partial order $\preceq$.



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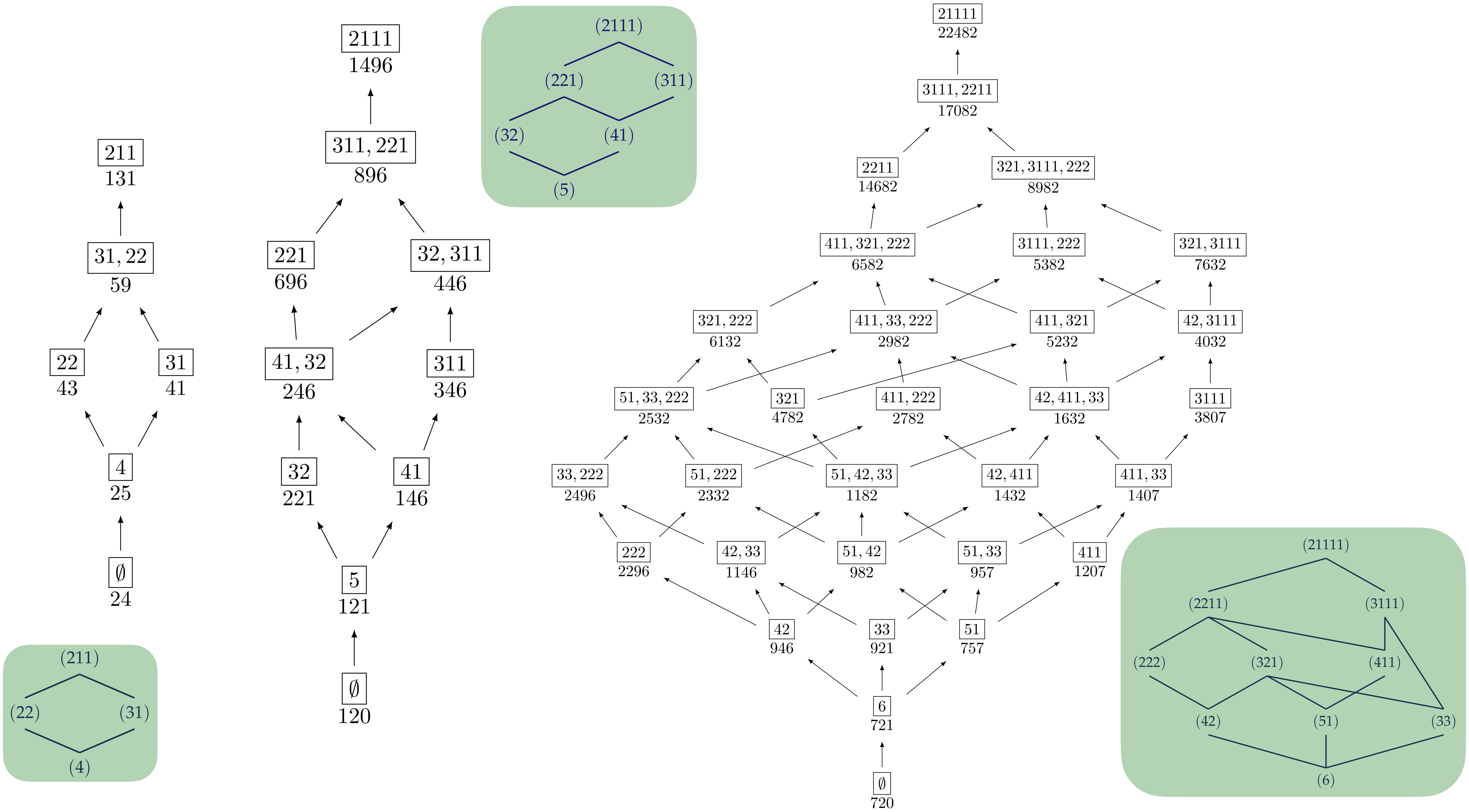
211
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146
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321, 3111, 222
8982
411, 321, 222
6582
3111, 222
5382
321, 3111
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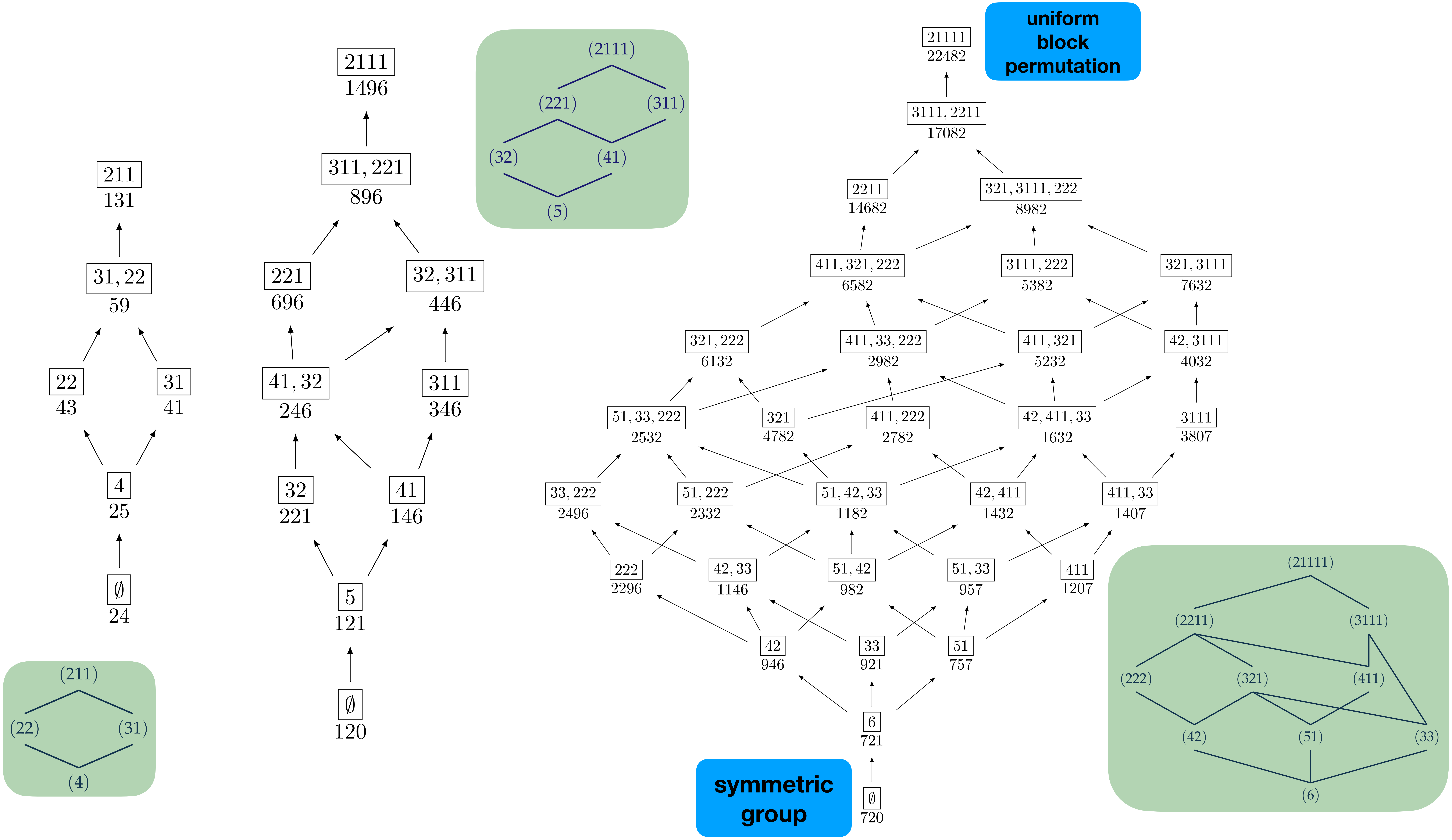
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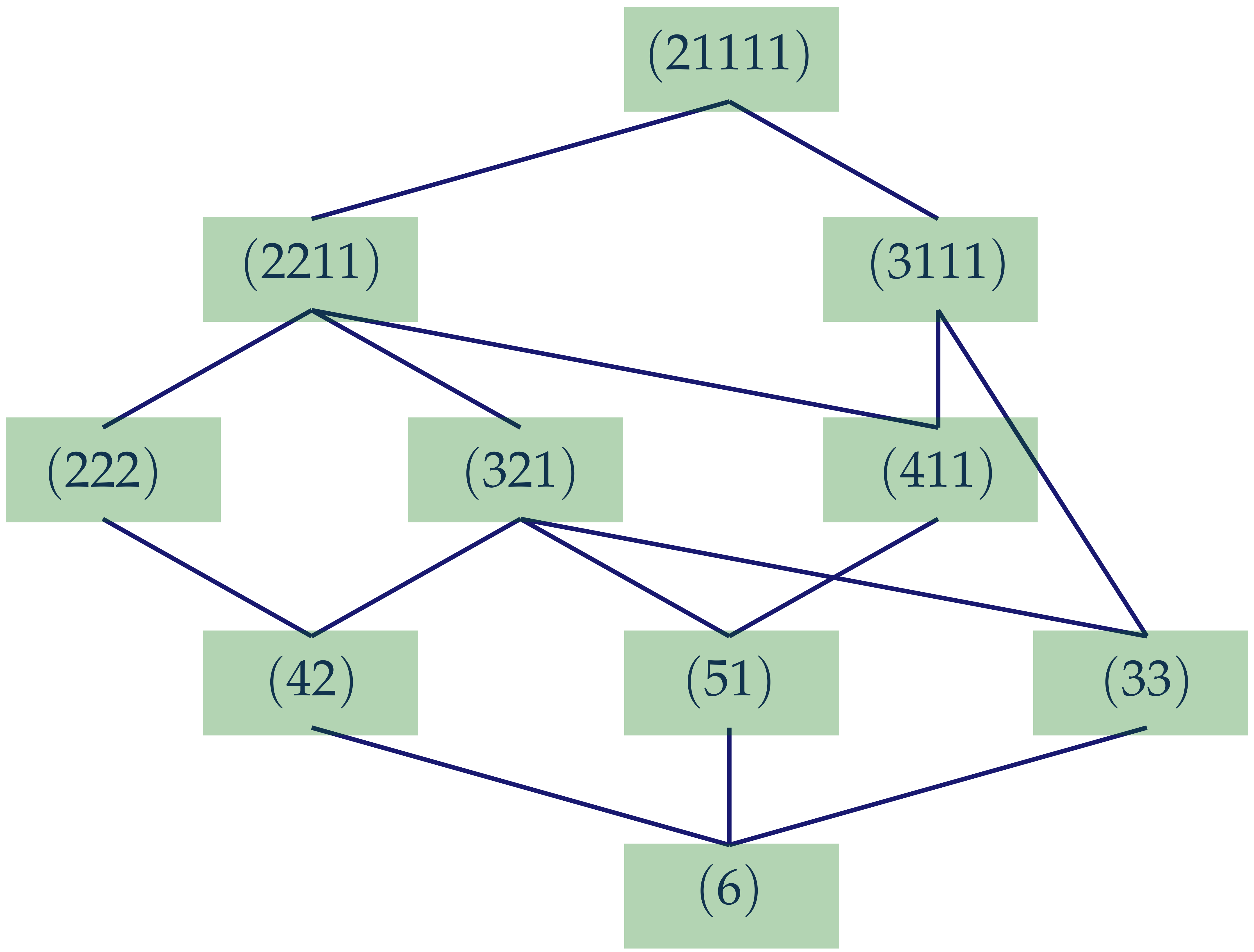
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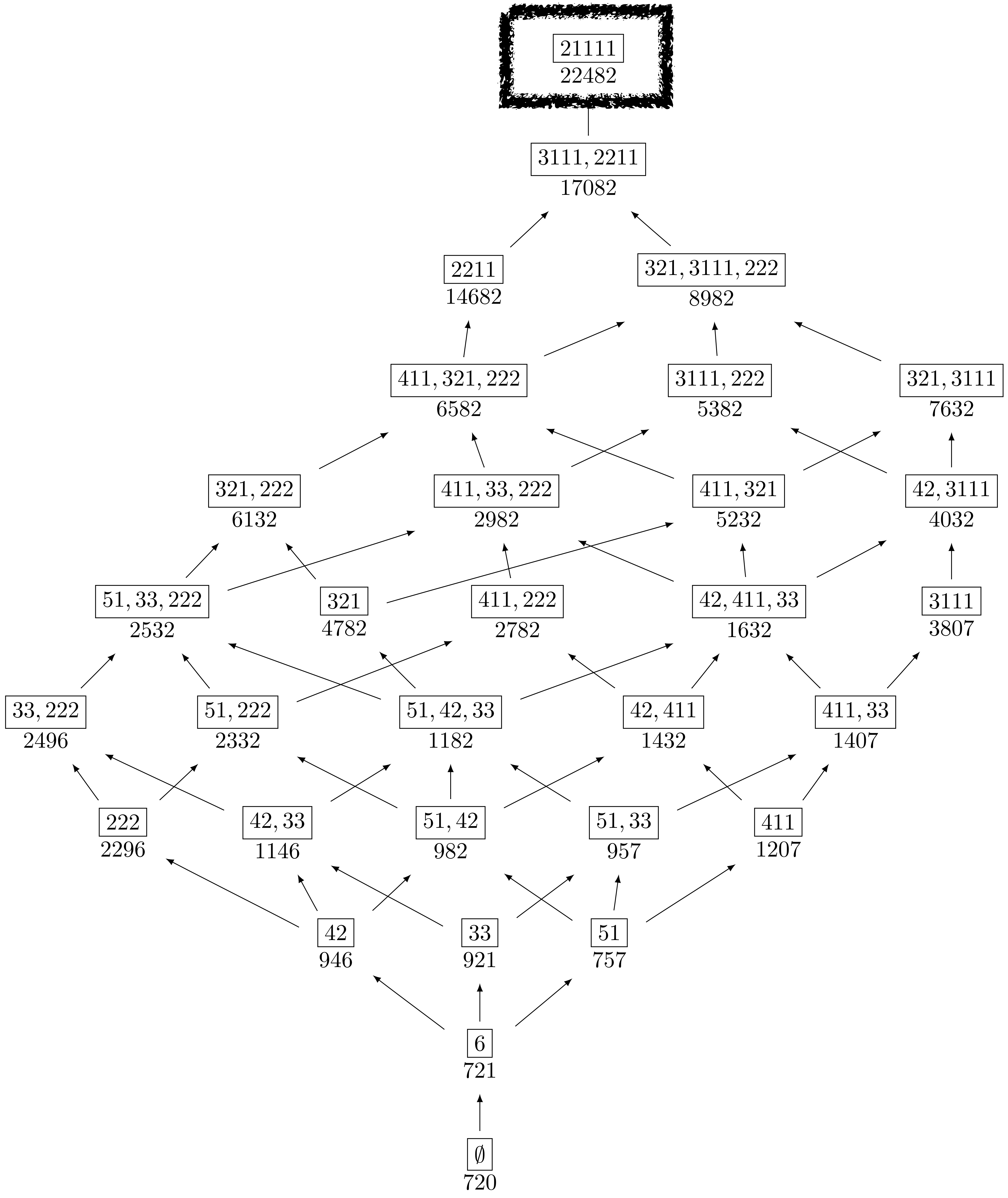
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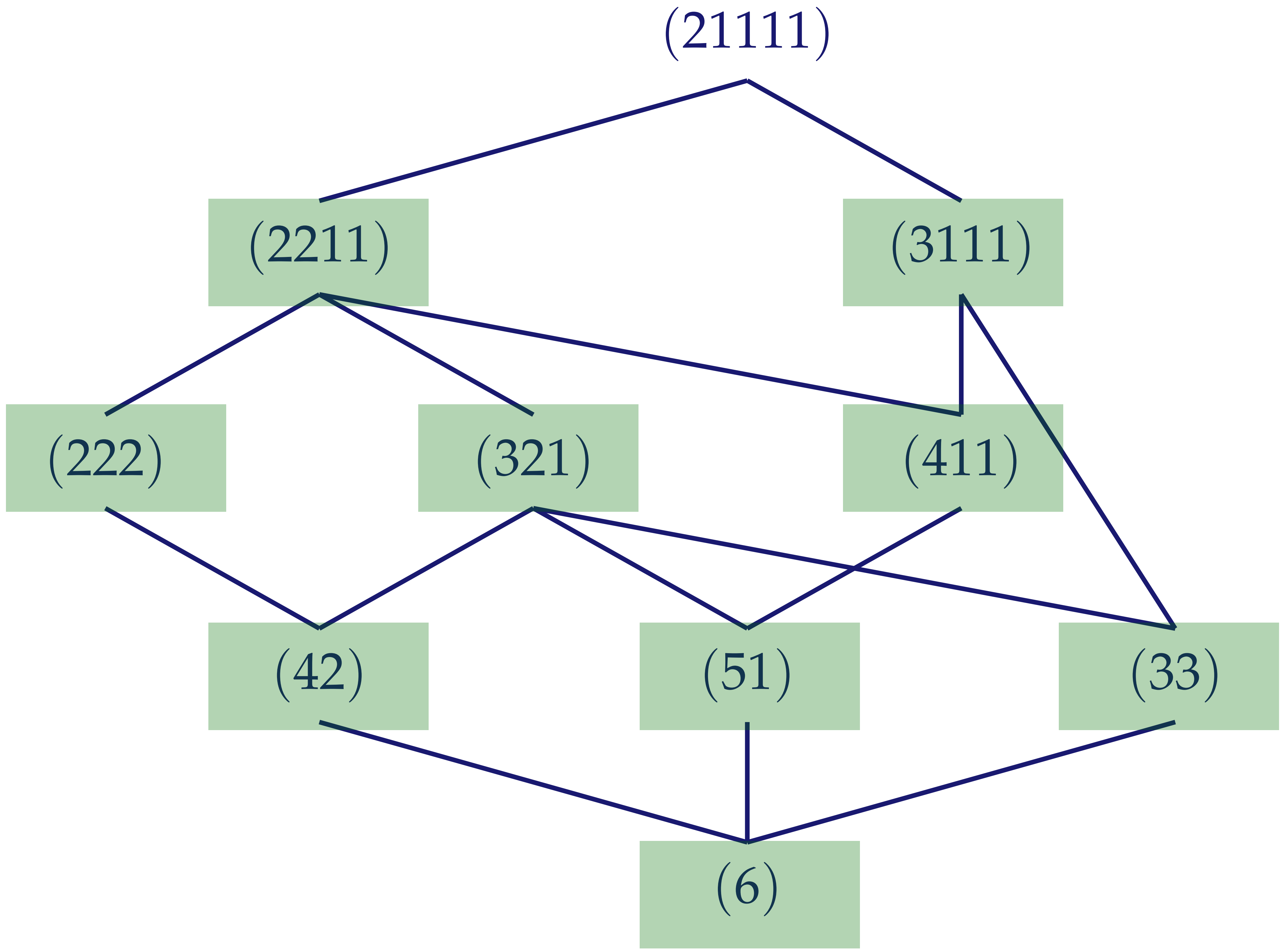
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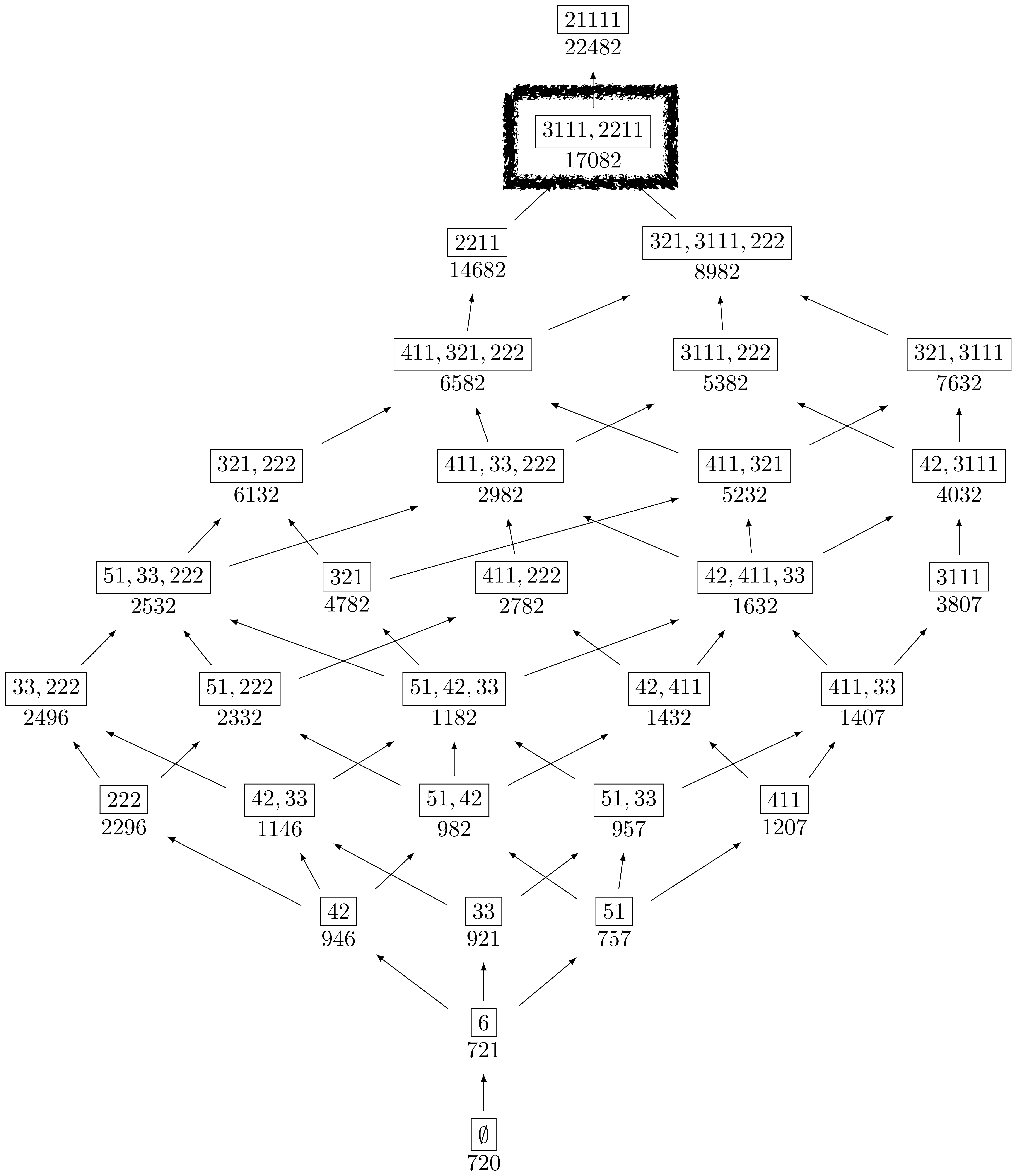
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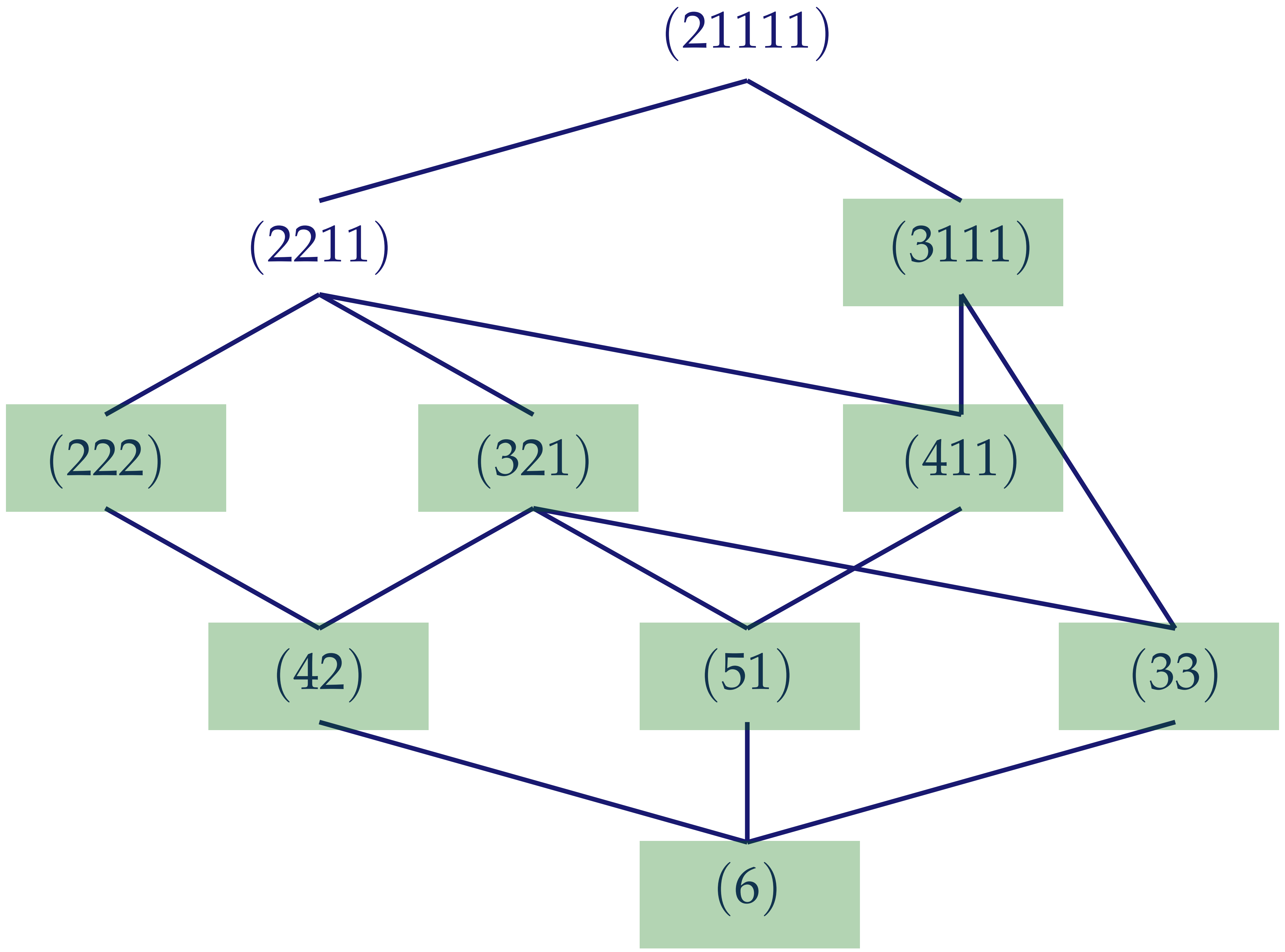
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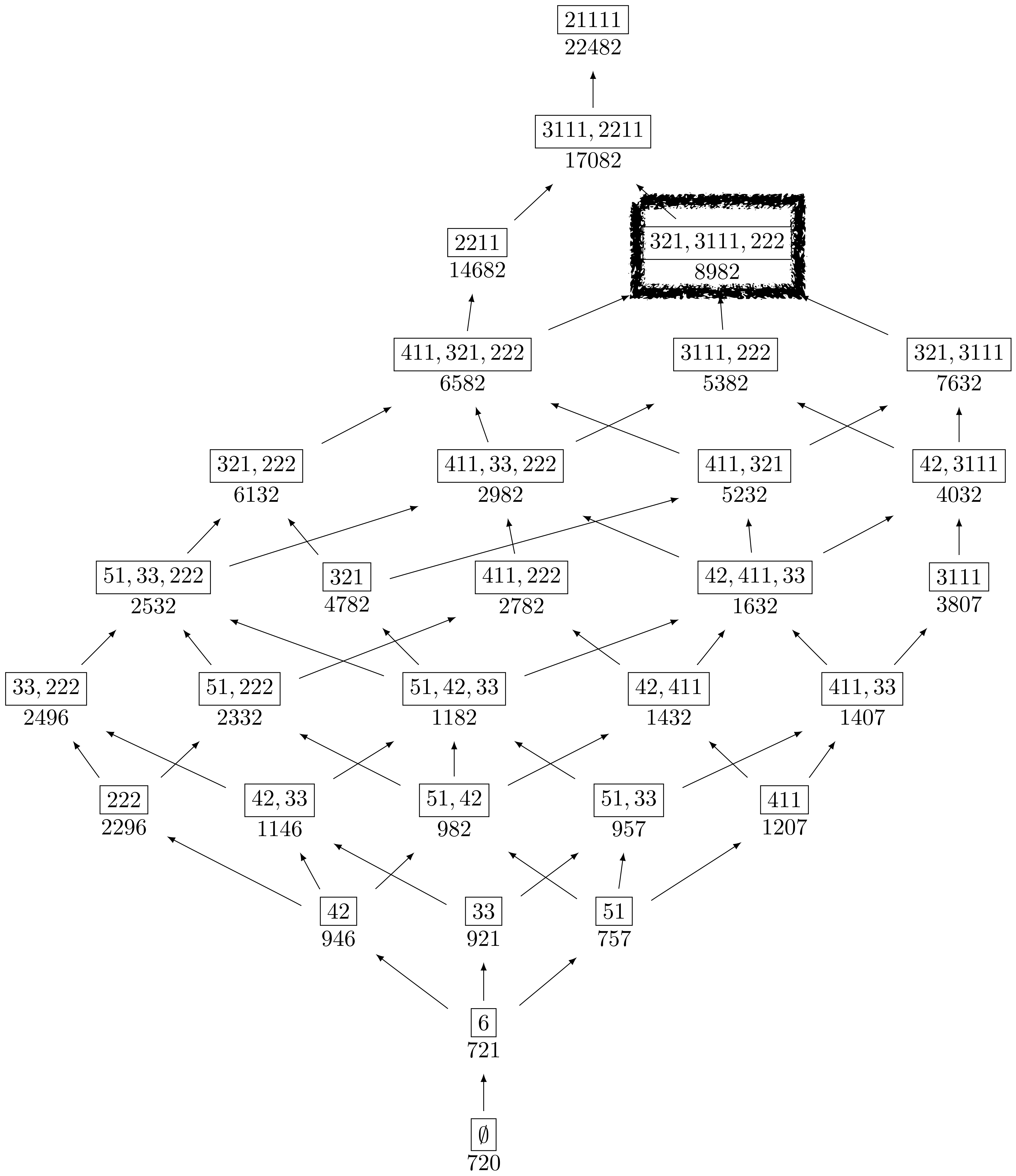
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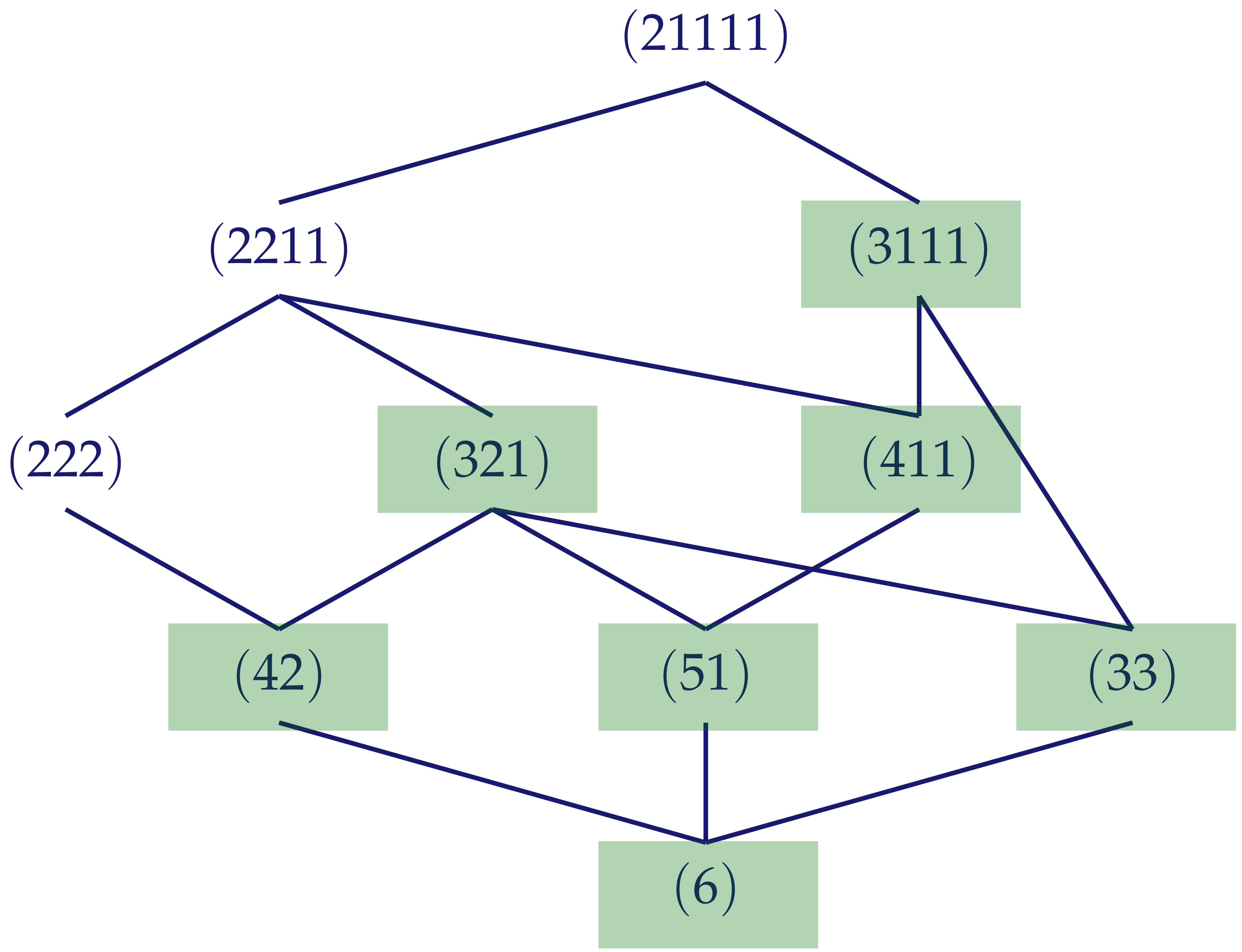
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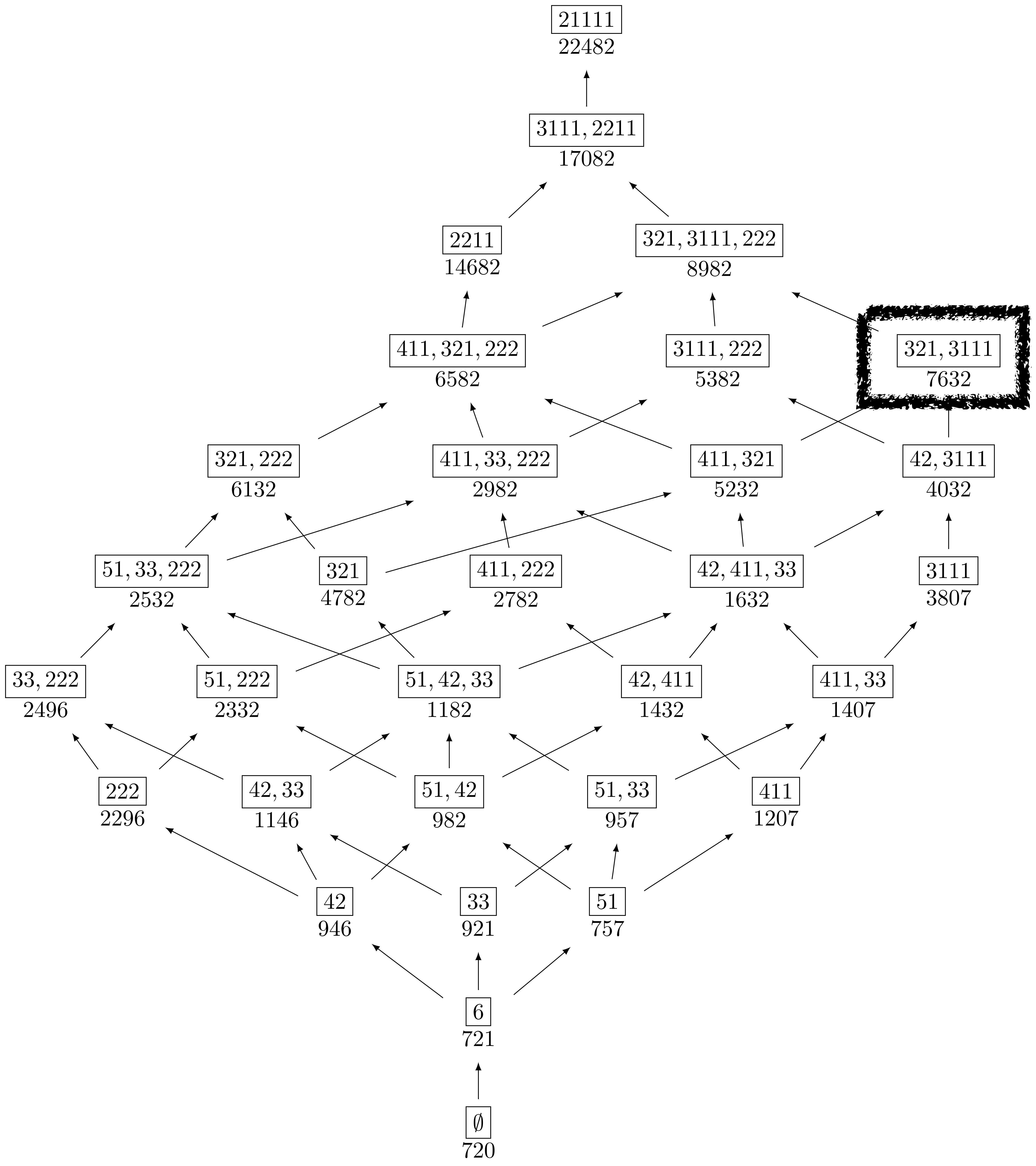
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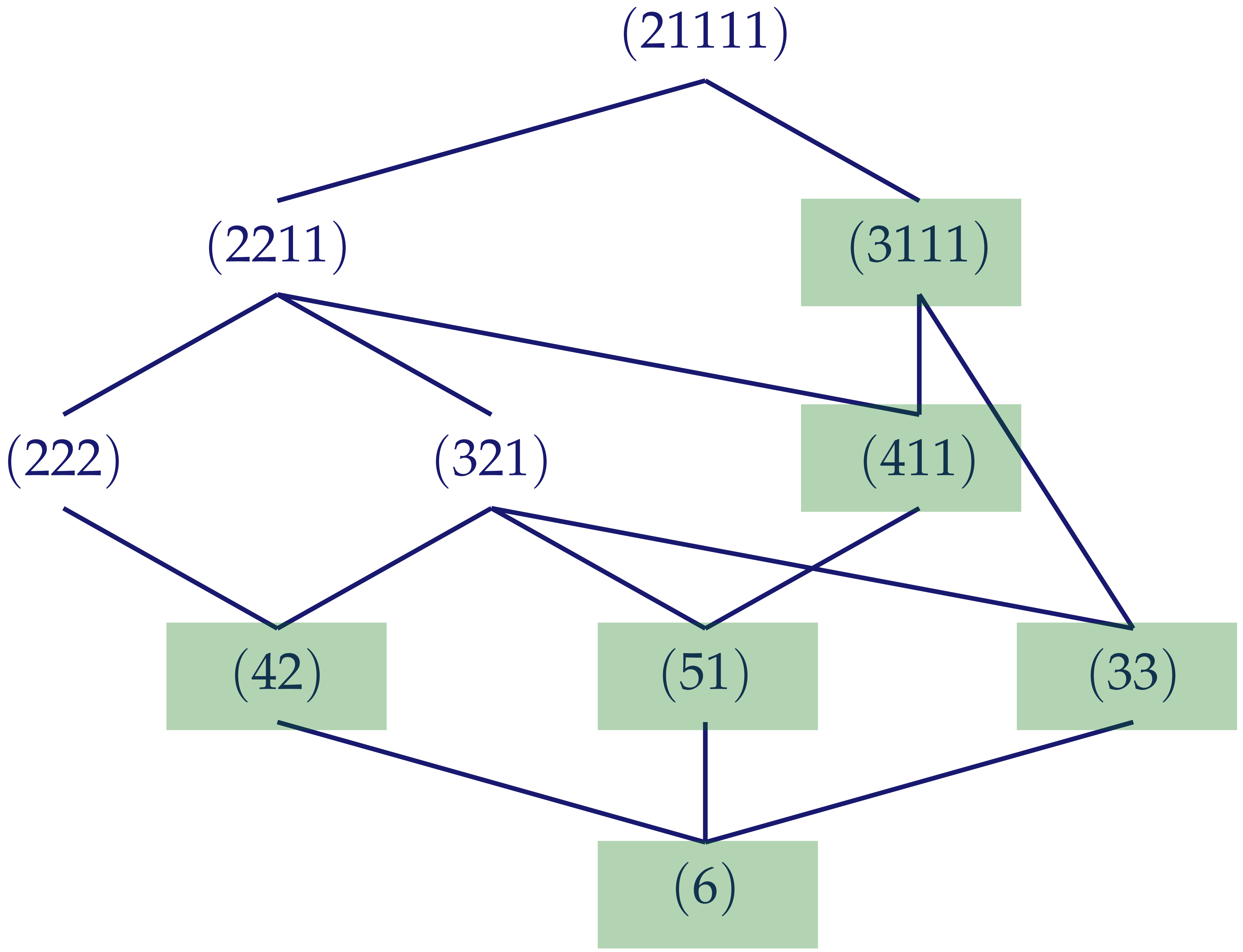
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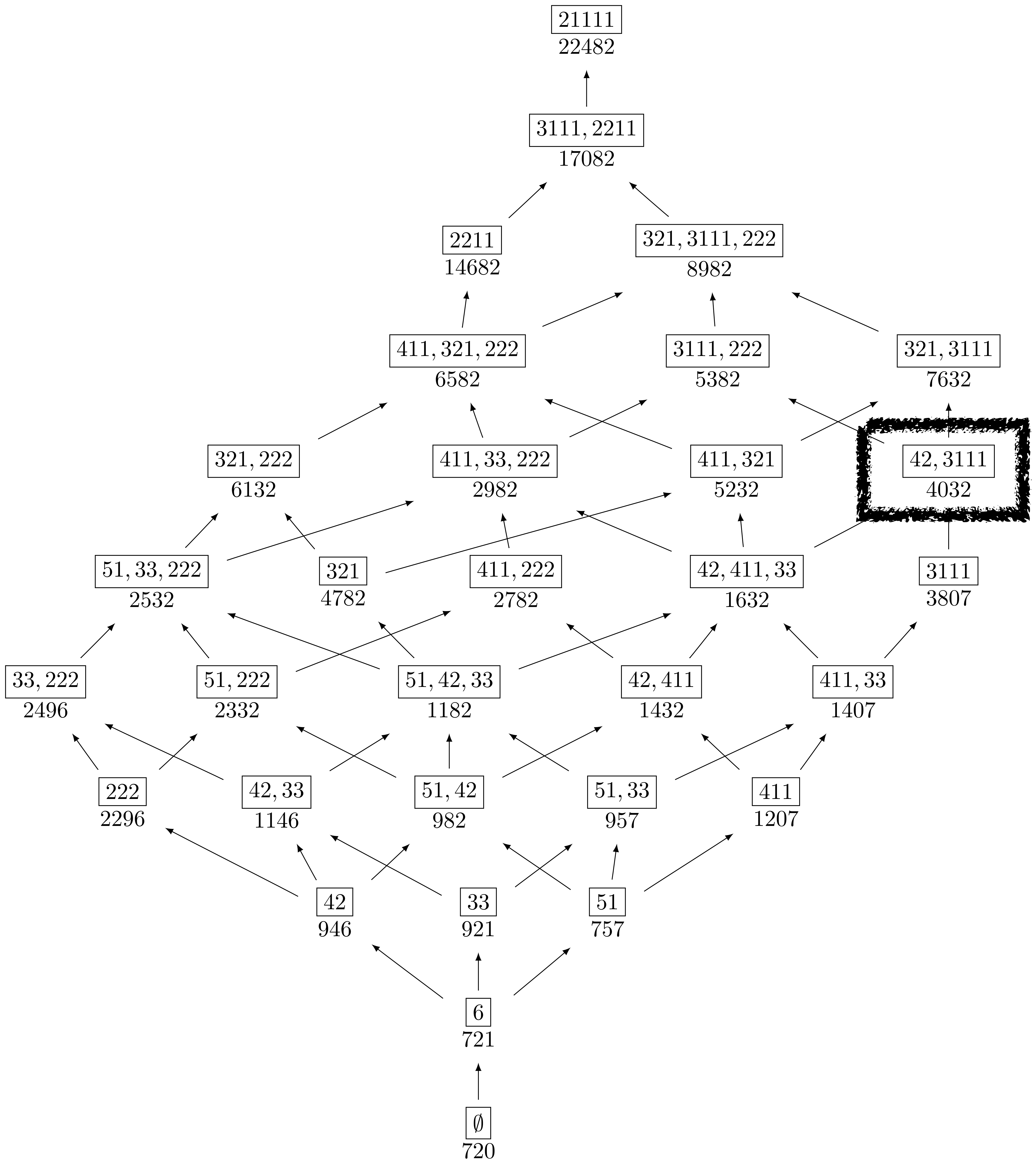
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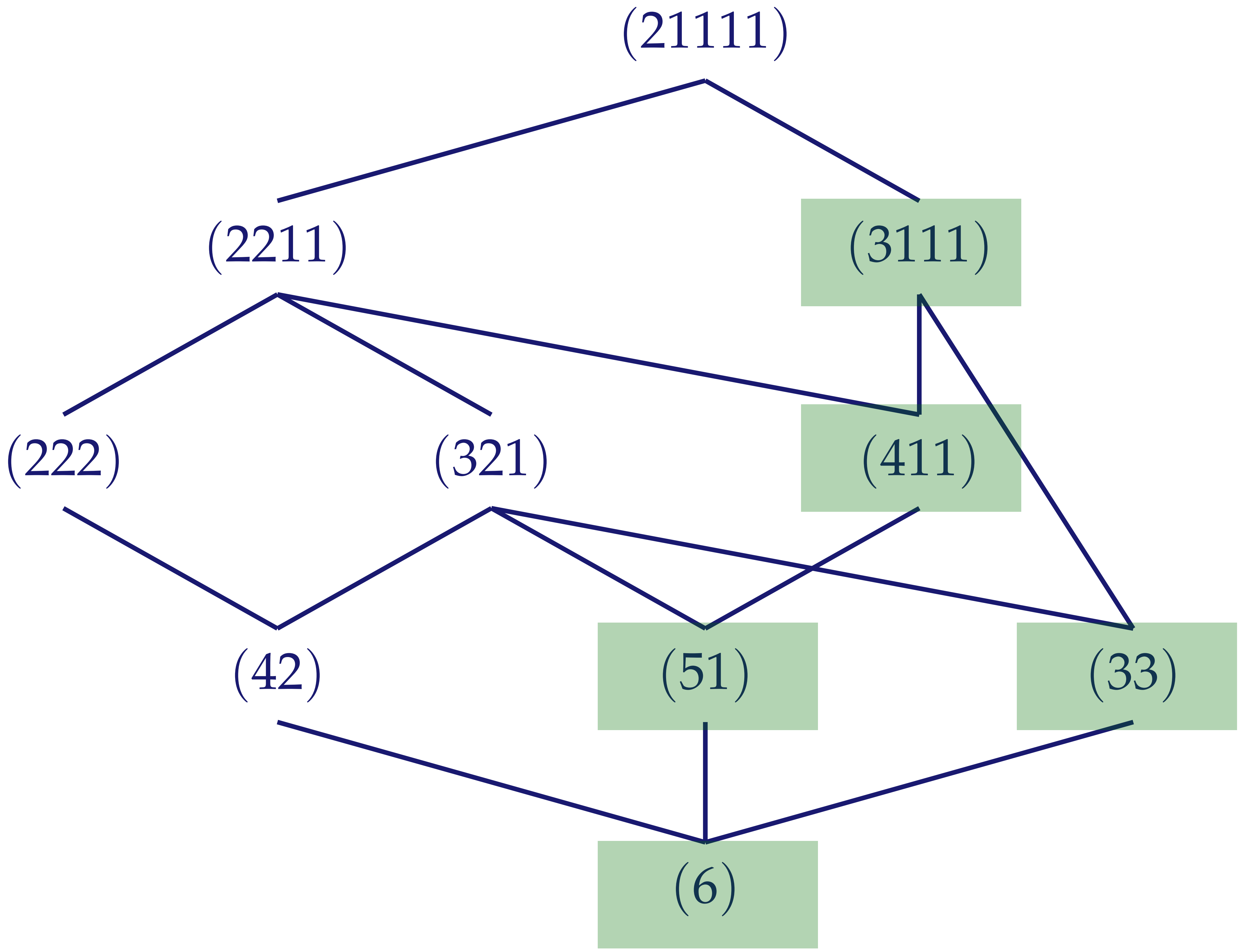
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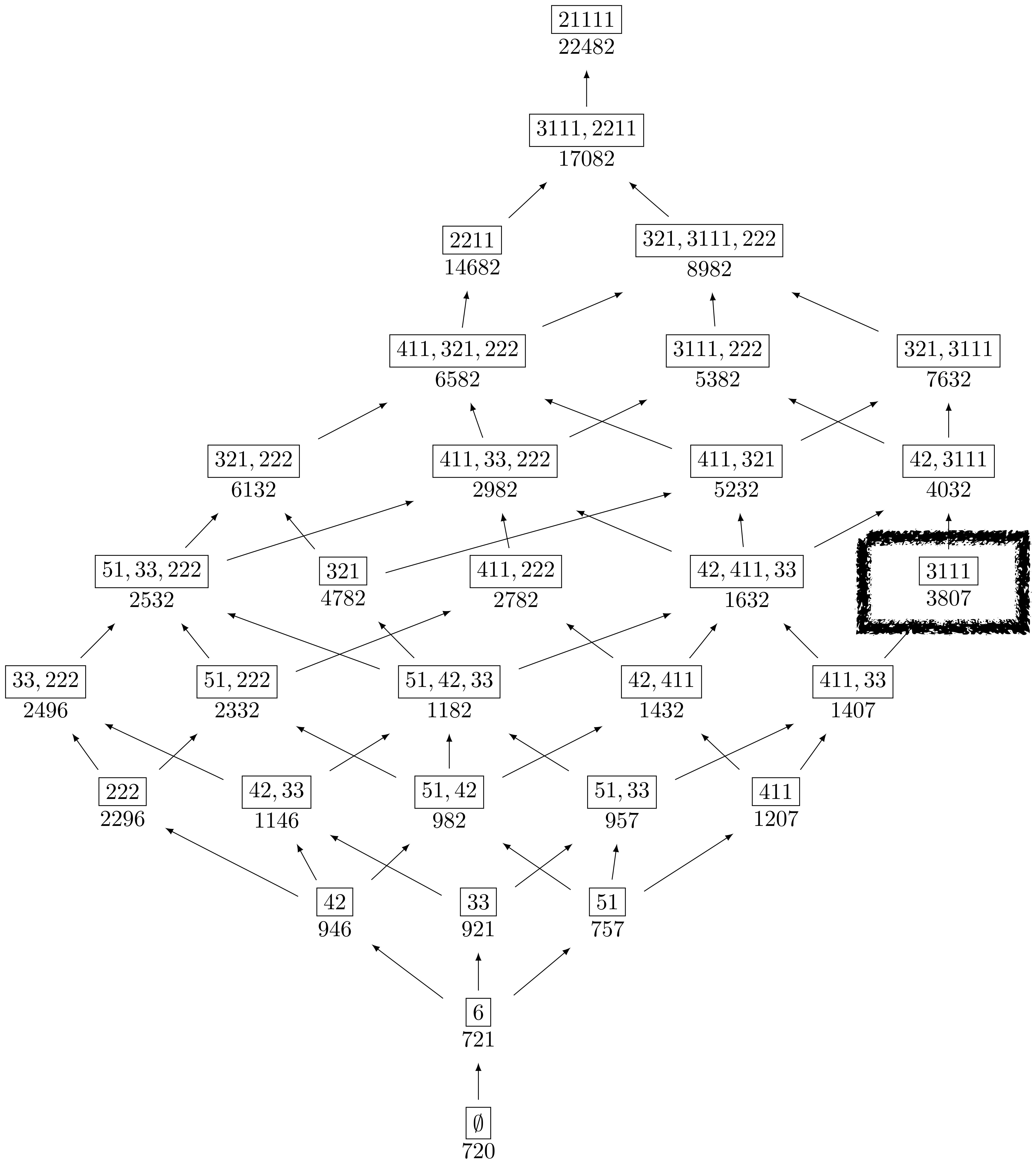
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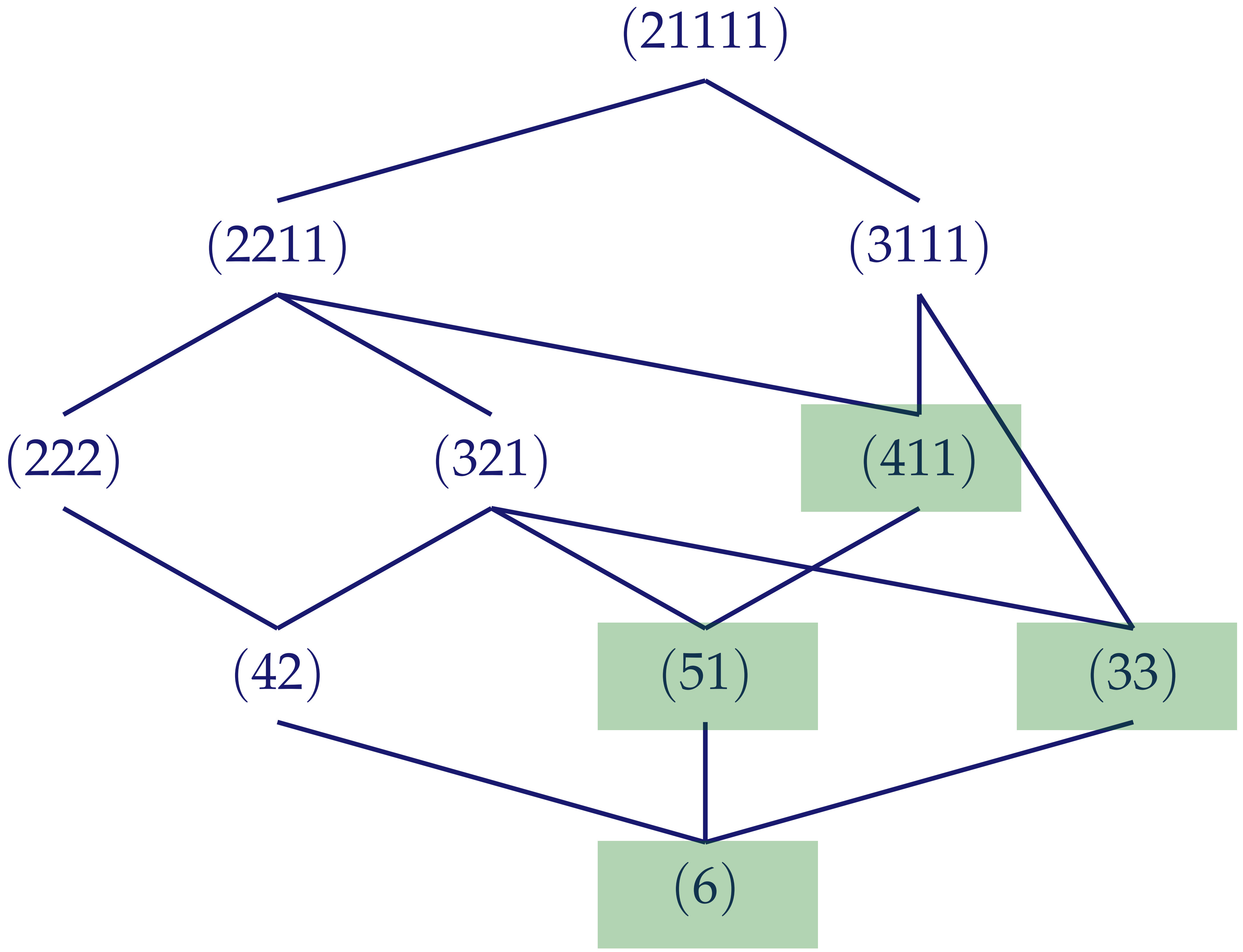
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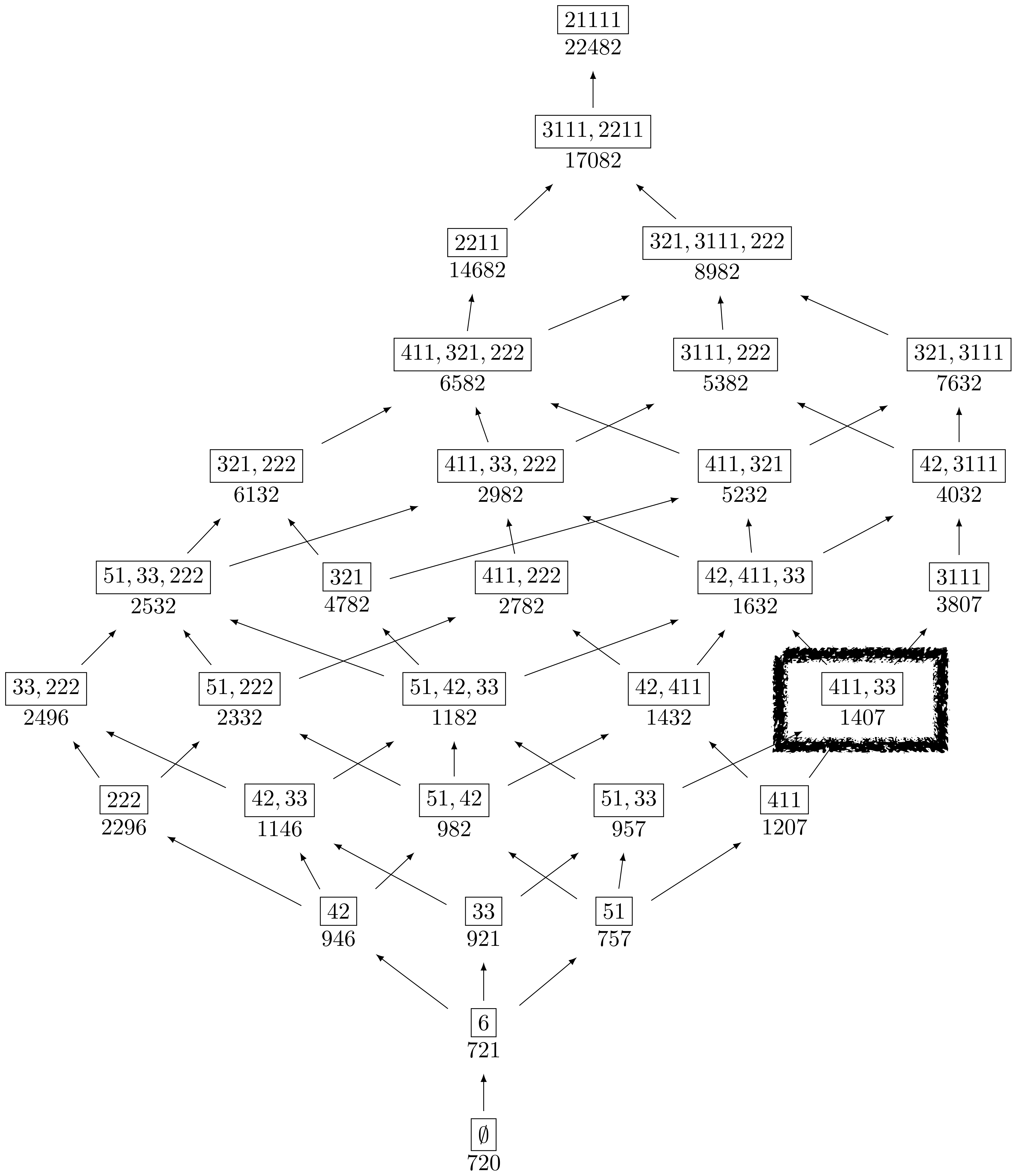
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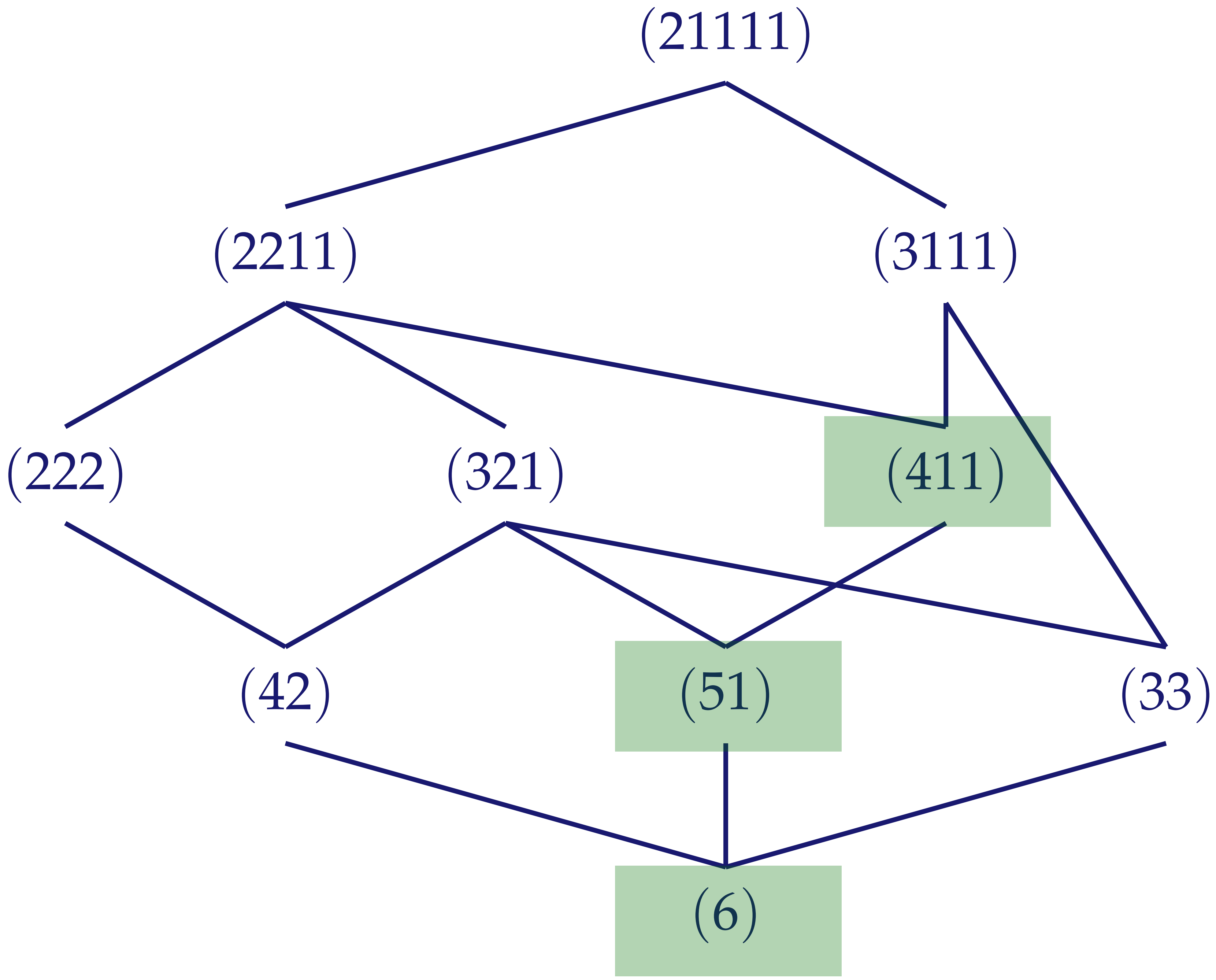
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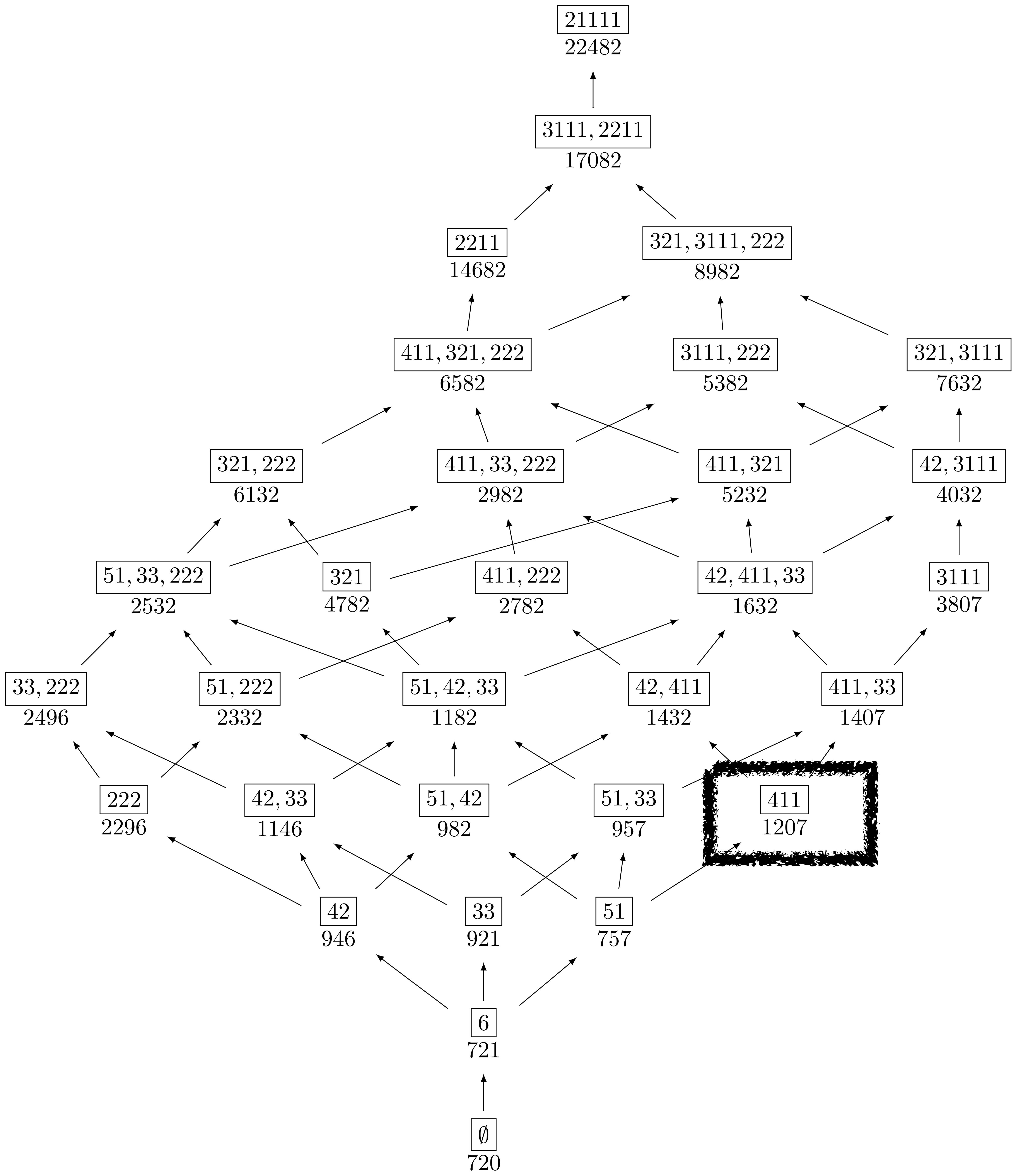
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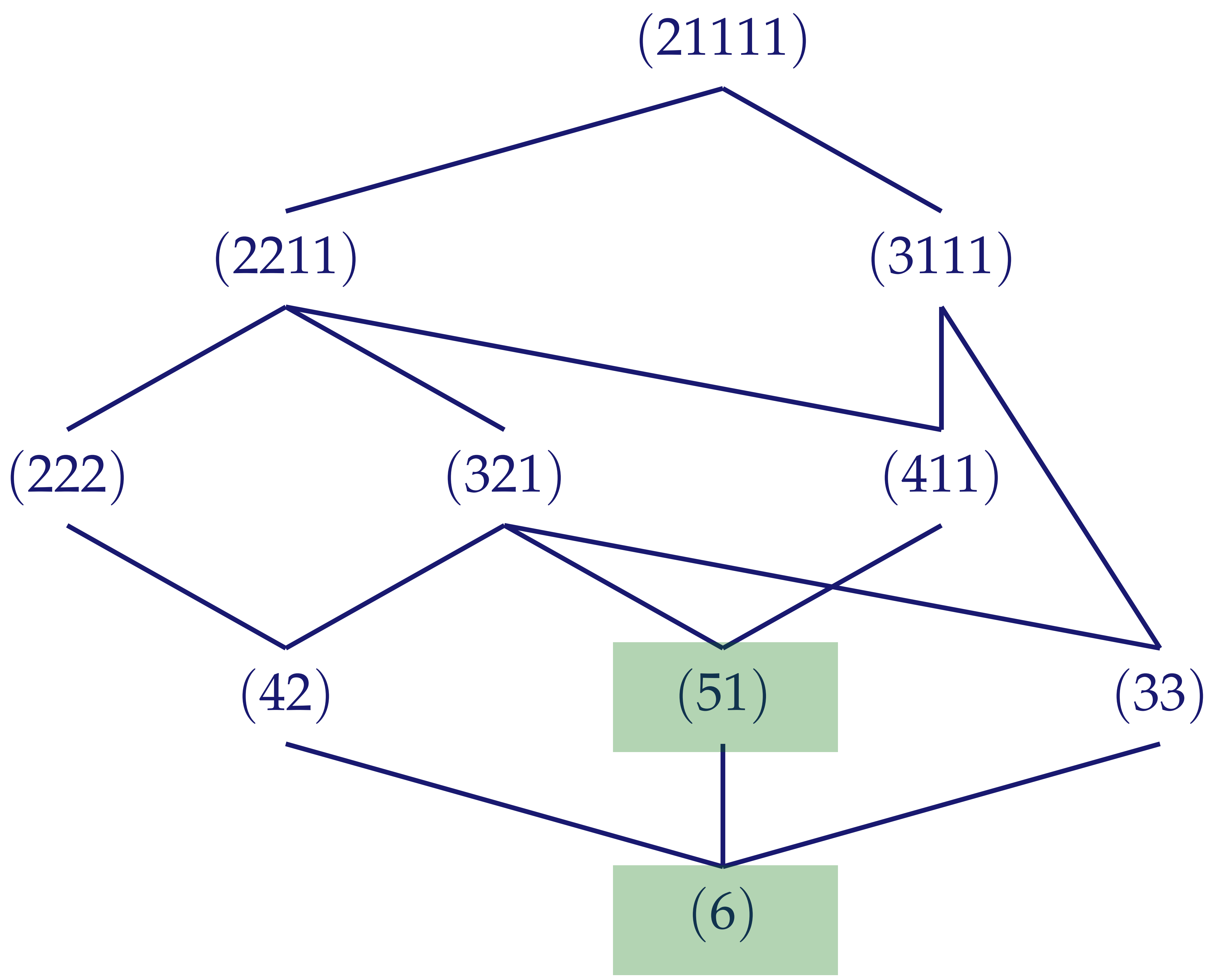
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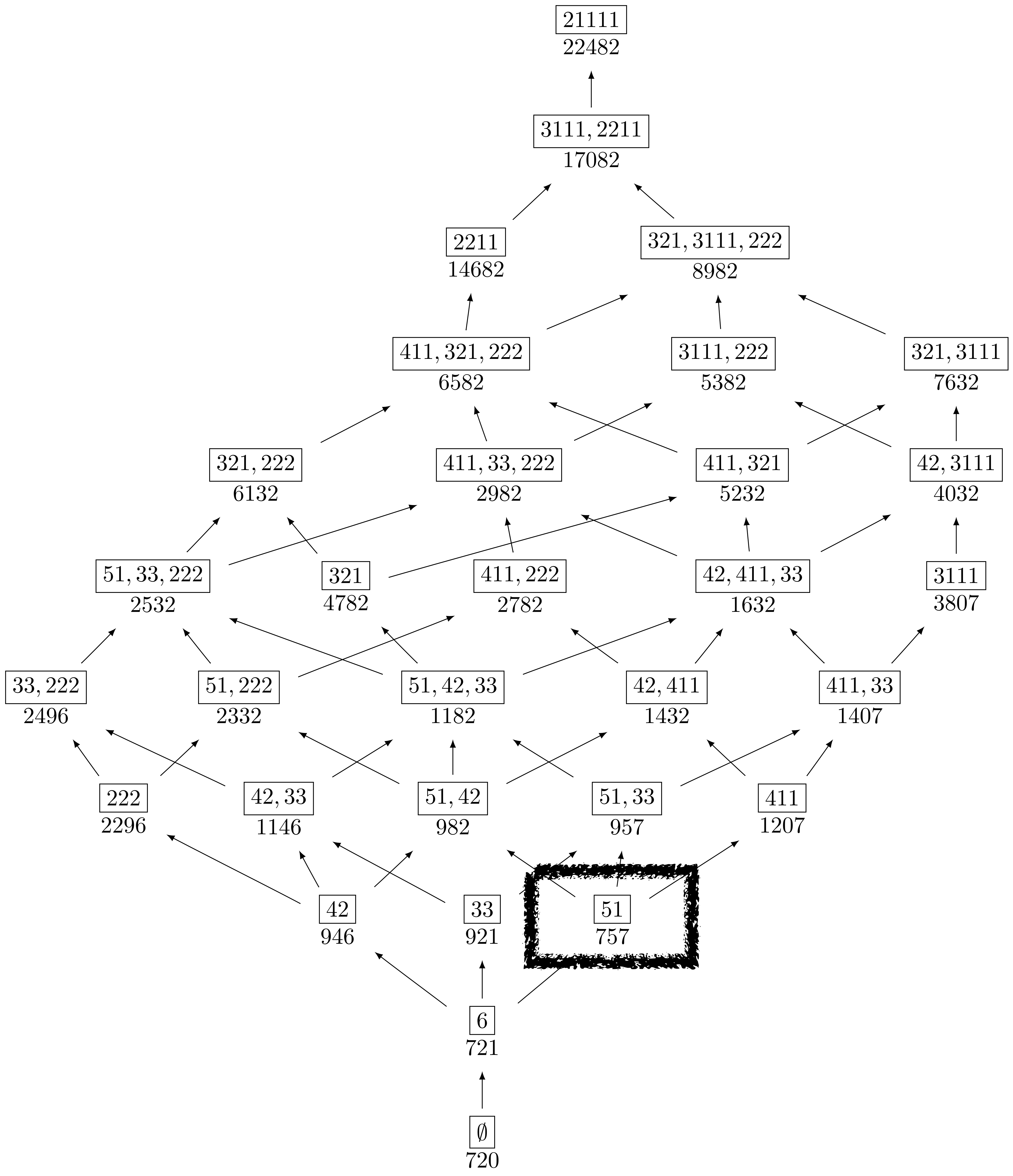
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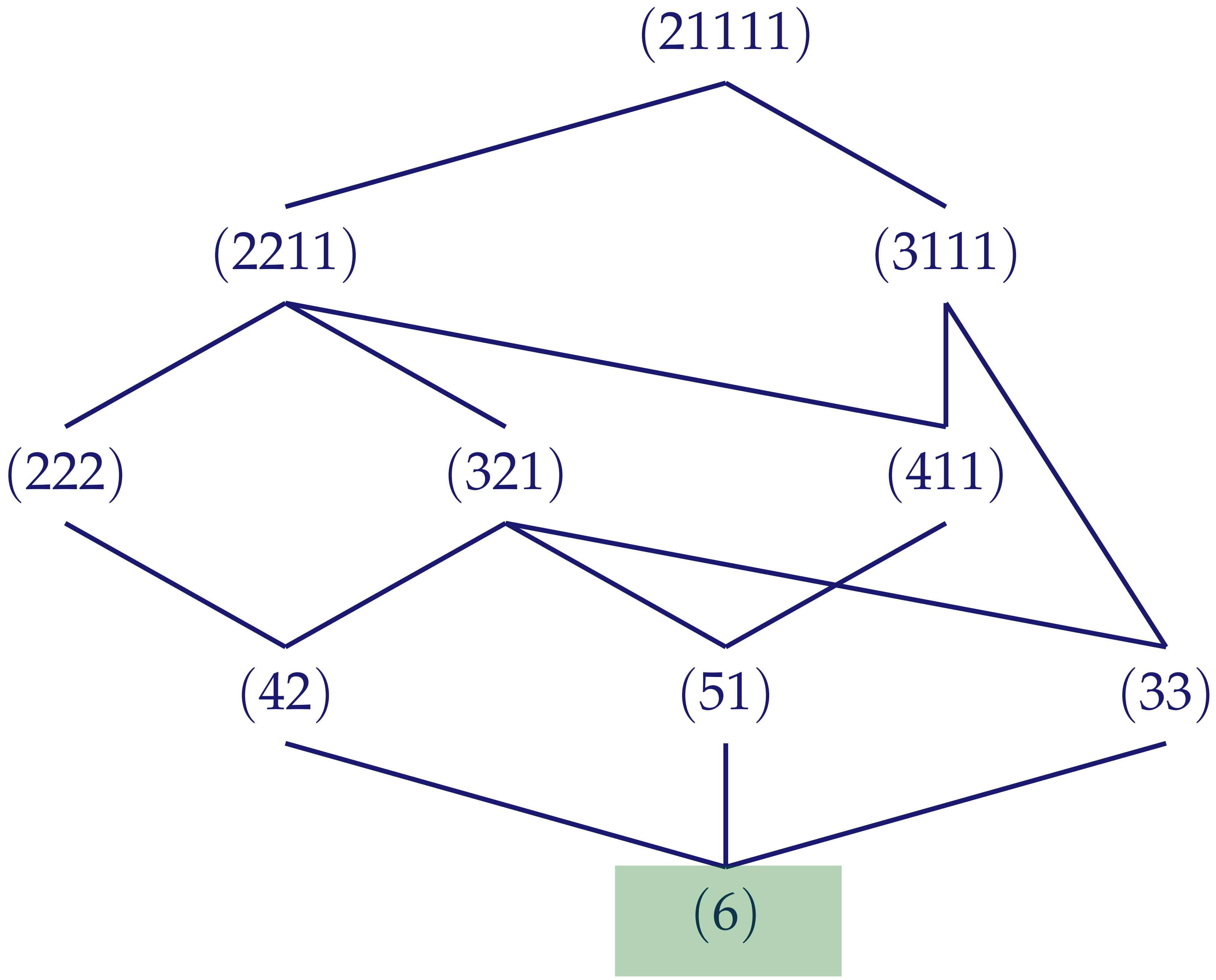
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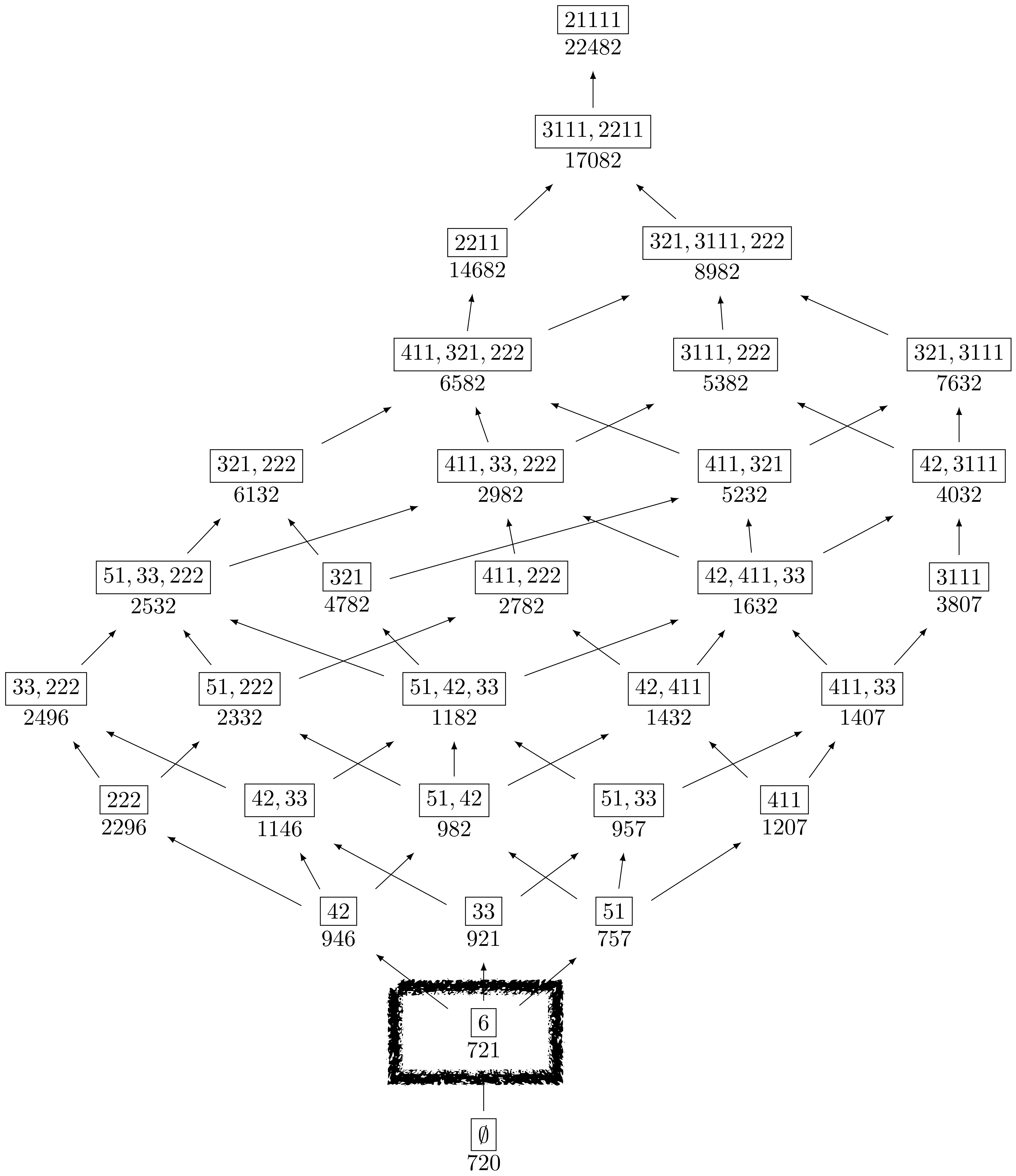
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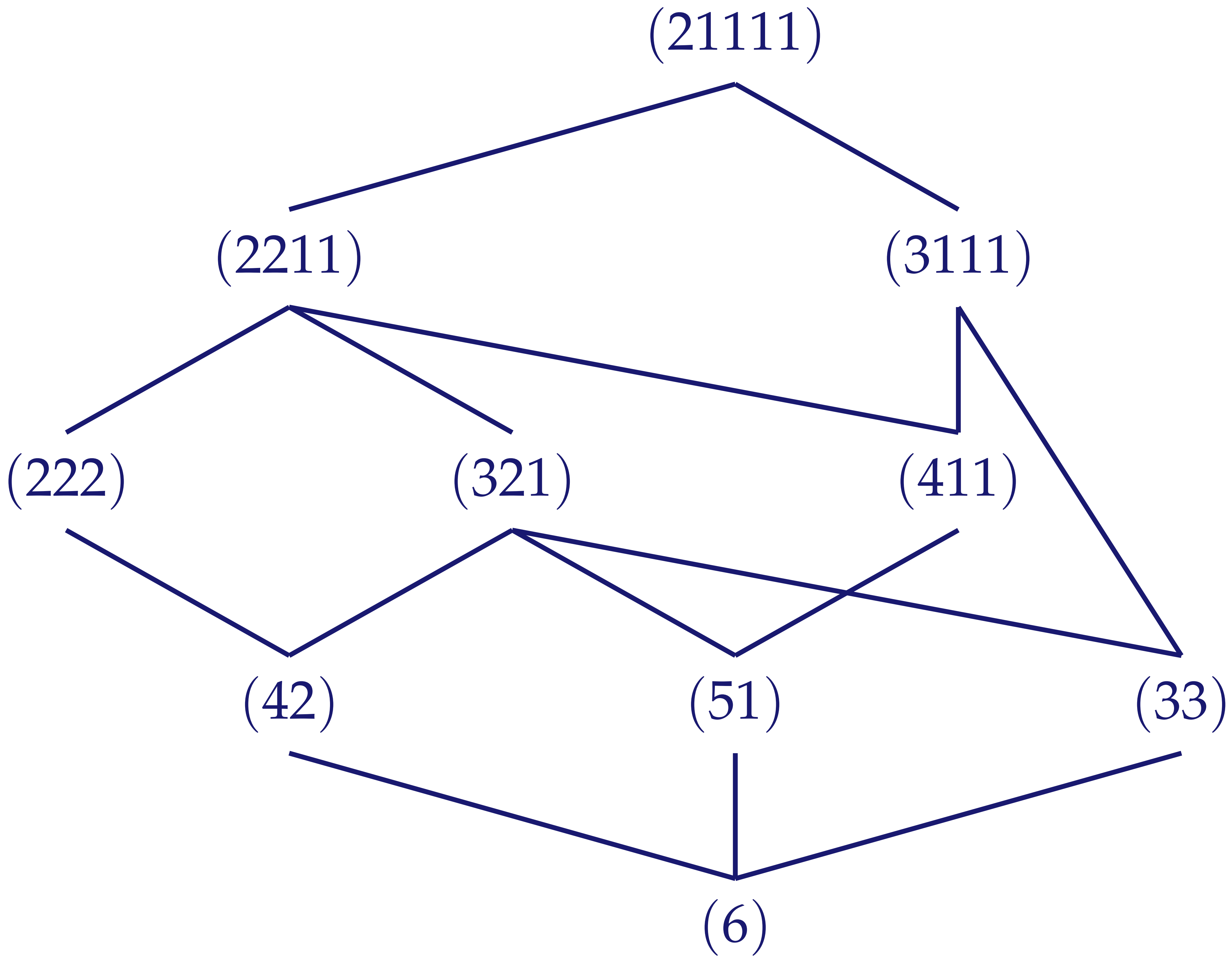
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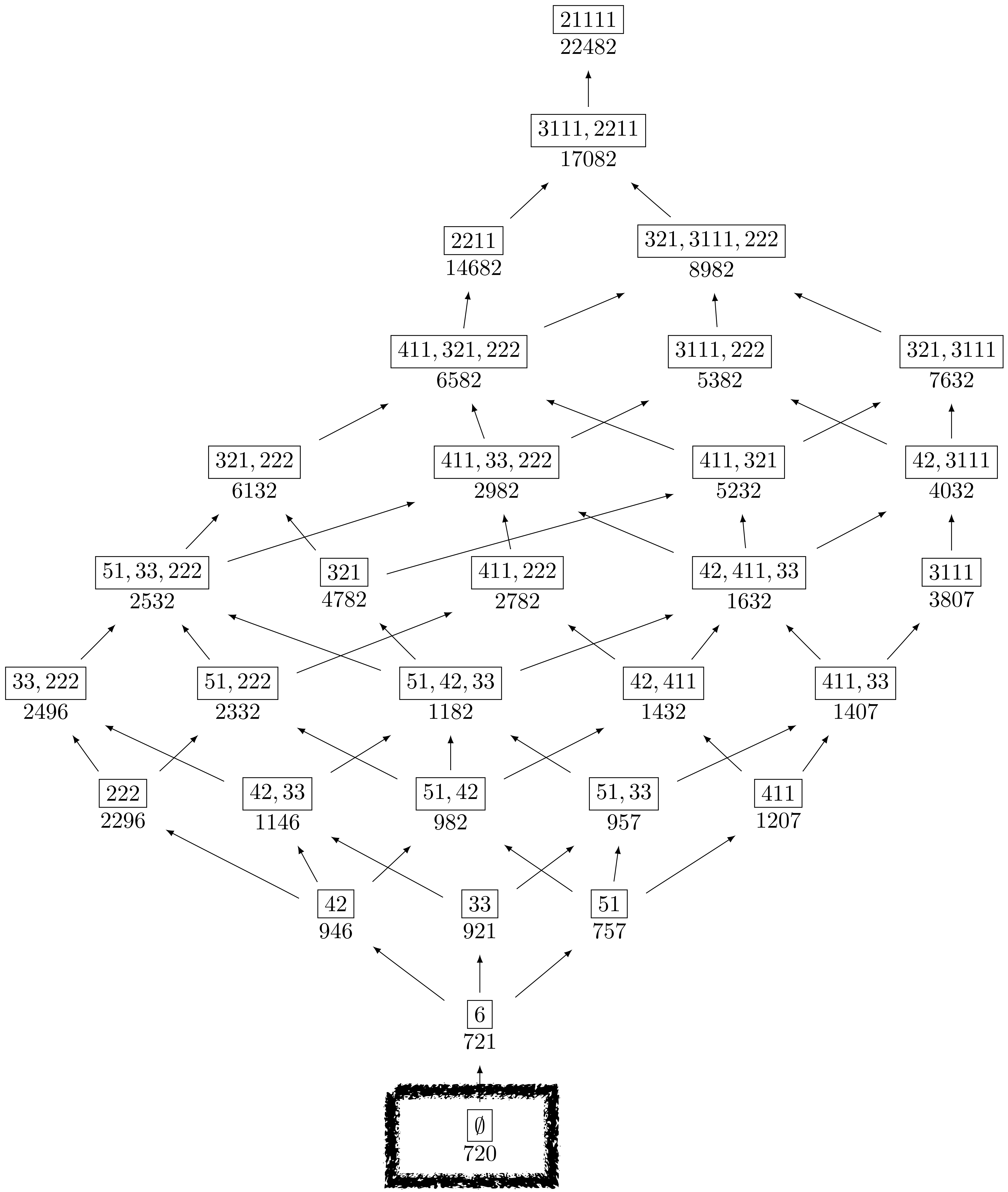
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Representation theory of uniform block partition sub-monoids

$\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ each $\lambda^{(i)}$ is a partition

$|\lambda^{(1)}| + 2|\lambda^{(2)}| + \cdots + k|\lambda^{(k)}| = k$

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$$\mathrm{Irreps}(\mathrm{UBP}_k^I) = \{\vec{\lambda} : (k^{|\lambda^{(k)}|}, \ldots, 2^{|\lambda^{(2)}|}, 1^{|\lambda^{(1)}|}) \in I \cup \{(1^k)\}\}$$

indexing set of the irreducibles of UBP_k^I

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$W^{\vec{\lambda}}_{\mathrm{UBP}_k^I}$ is spanned by standard vector tableaux of shape $\vec{\lambda}$

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Example:

typical basis element

$\vec{\lambda} = ((2,1), (2,2), (1,1))$

$$\mathbf{T} = \left(\begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 67 & ab \\ \hline 45 & 89 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline fgh \\ \hline cde \\ \hline \end{array} \right)$$

$$\overrightarrow{\mathsf{type}}(\vec{\lambda}) = (k^{|\lambda^{(k)}|}, \ldots, 2^{|\lambda^{(2)}|}, 1^{|\lambda^{(1)}|})$$

Theorem: (Orellana-Saliola-Schilling-Z '25)

(1) If $\overrightarrow{\mathsf{type}}(\vec{\lambda}) \in I$, *then* $\mathrm{Res}^{\mathrm{UBP}_k}_{\mathrm{UBP}^I_k}\left(W^{\vec{\lambda}}_{\mathrm{UBP}_k}\right)$ *is irreducible and*

$$\mathrm{Res}^{\mathrm{UBP}_k}_{\mathrm{UBP}^I_k}\left(W^{\vec{\lambda}}_{\mathrm{UBP}_k}\right) \cong W^{\vec{\lambda}}_{\mathrm{UBP}^I_k}.$$

(2) If there exists an $r \in [k]$ *such that*

$$\left(1^{(\sum_{i=1}^{r-2} i|\lambda^{(i)}|)},\ (r-1)^{|\lambda^{(r-1)}|},\ r^{|\lambda^{(r)}|},\ \ldots,\ k^{|\lambda^{(k)}|}\right) \notin I$$

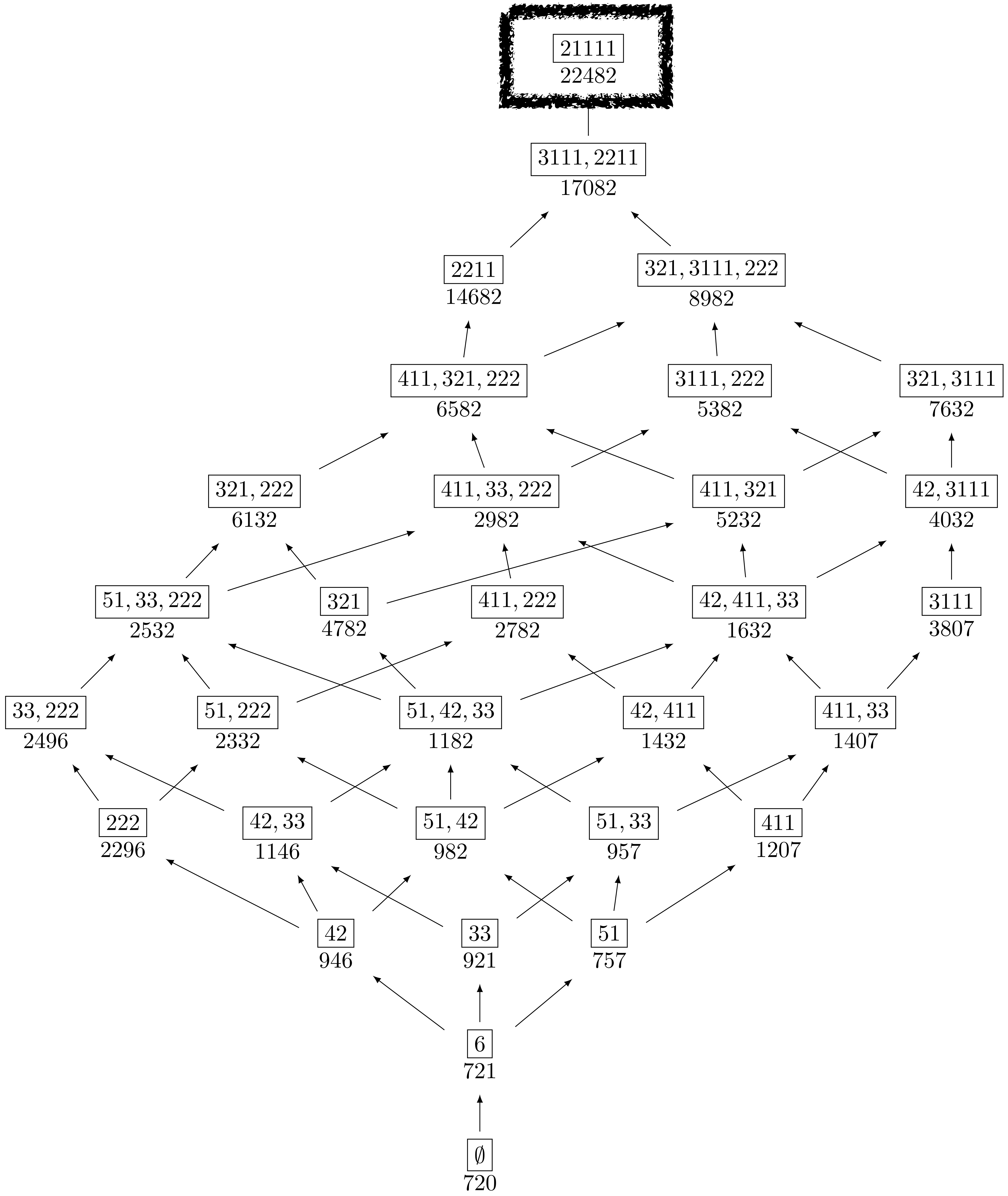
$$\left(1^{(\sum_{i=1}^{r-2} i|\lambda^{(i)}|)+(r-1)|\lambda^{(r-1)}|},\ r^{|\lambda^{(r)}|},\ \ldots,\ k^{|\lambda^{(k)}|}\right) \in I$$

then

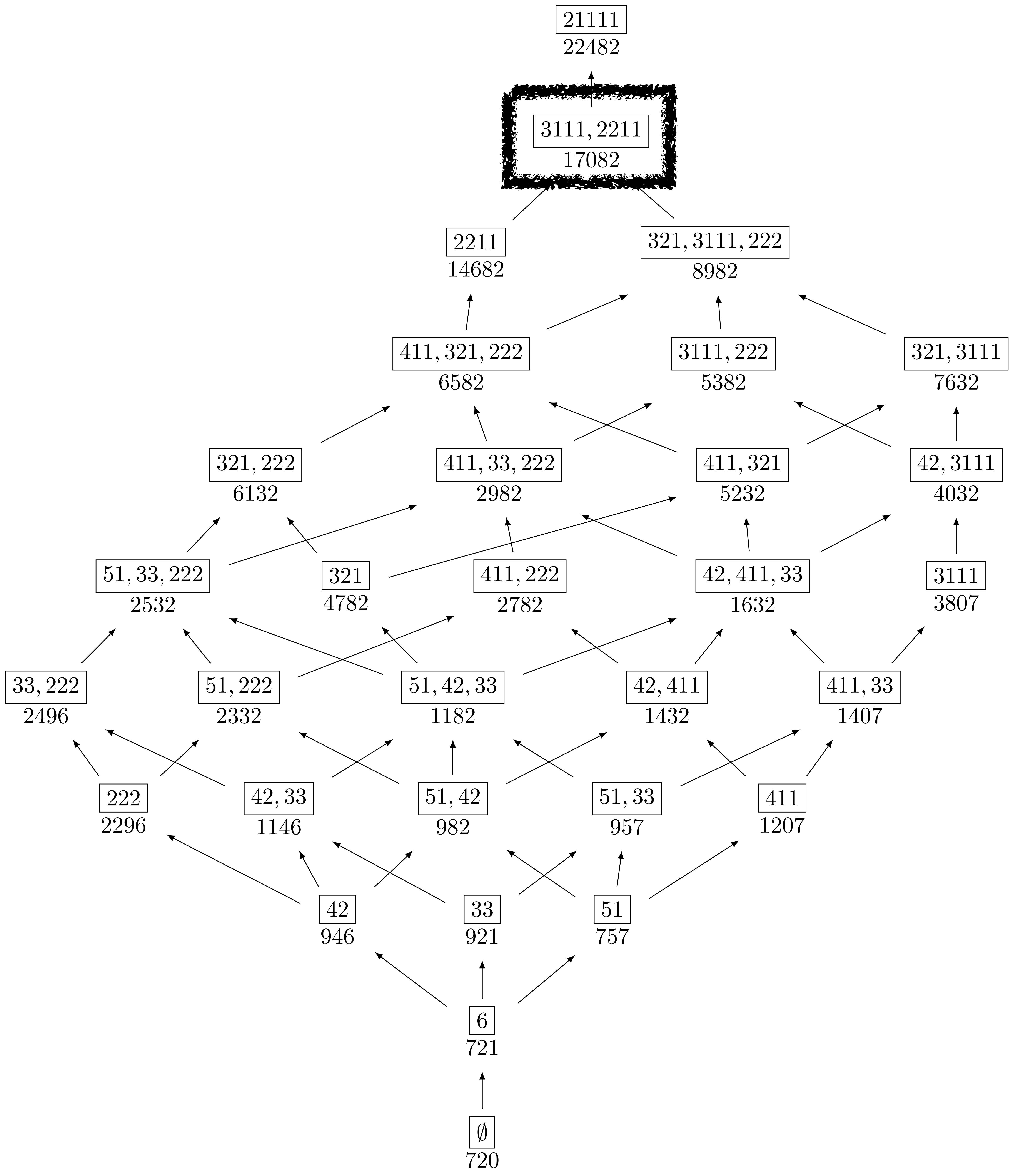
$$\mathrm{Res}^{\mathrm{UBP}_k}_{\mathrm{UBP}^I_k}\left(W^{\vec{\lambda}}_{\mathrm{UBP}_k}\right) \cong \bigoplus_{\mu \vdash k}\left(W^{\left(\mu,\varnothing,\ldots,\varnothing,\lambda^{(r)},\lambda^{(r+1)},\ldots,\lambda^{(k)}\right)}_{\mathrm{UBP}^I_k}\right)^{\oplus a_{\mu,\left(\lambda^{(1)},\ldots,\lambda^{(r-1)}\right)}}$$

where $a_{\mu\vec{\lambda}} := \langle s_\mu, s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \cdots s_{\lambda^{(\ell)}}[s_\ell] \rangle$

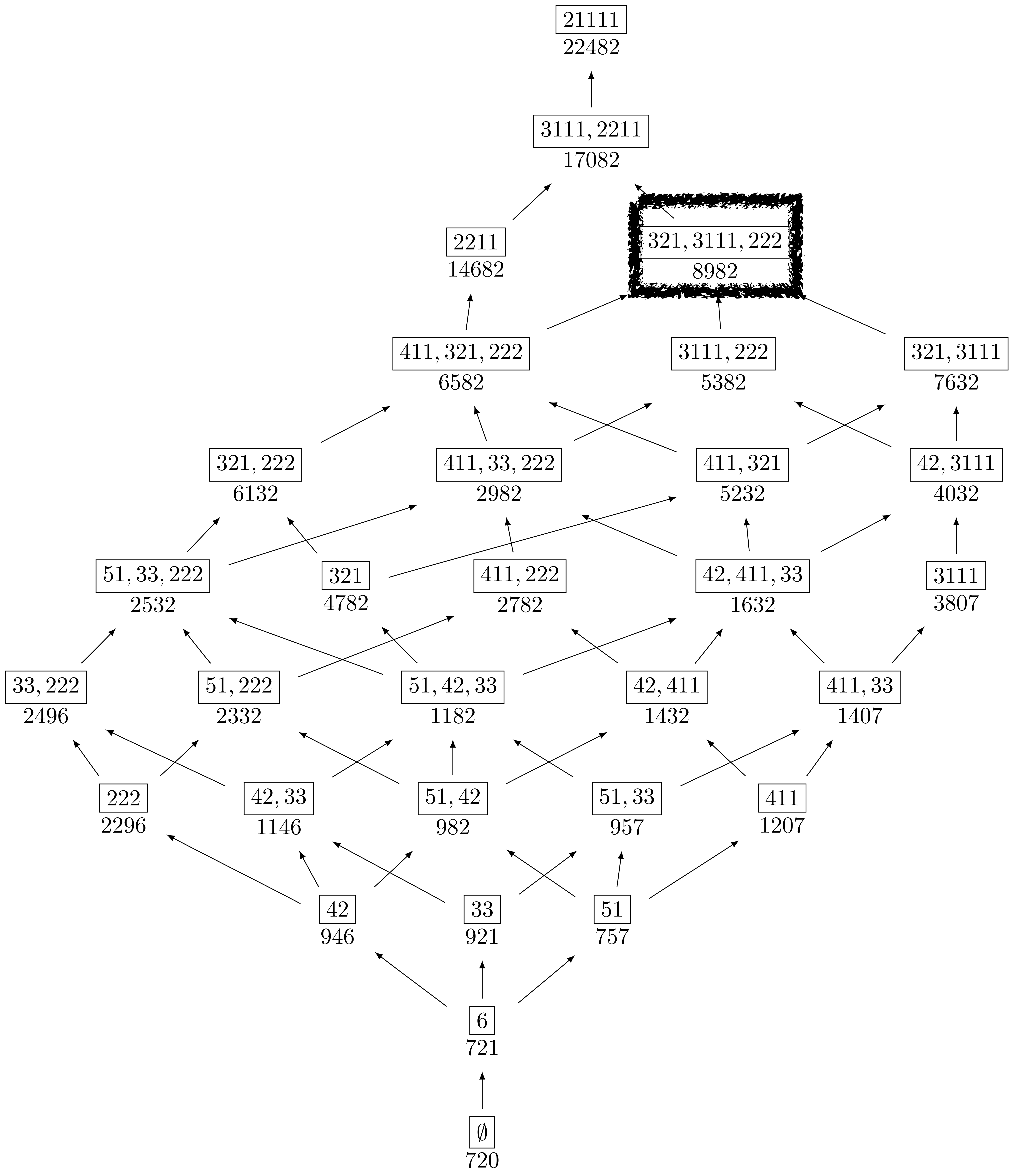
$$\mathrm{Res}^{U_6}_{U_6^I} W_{U_6}^{(\square,\square,\square)} \cong W_{U_6^I}^{(\square,\square,\square)}$$



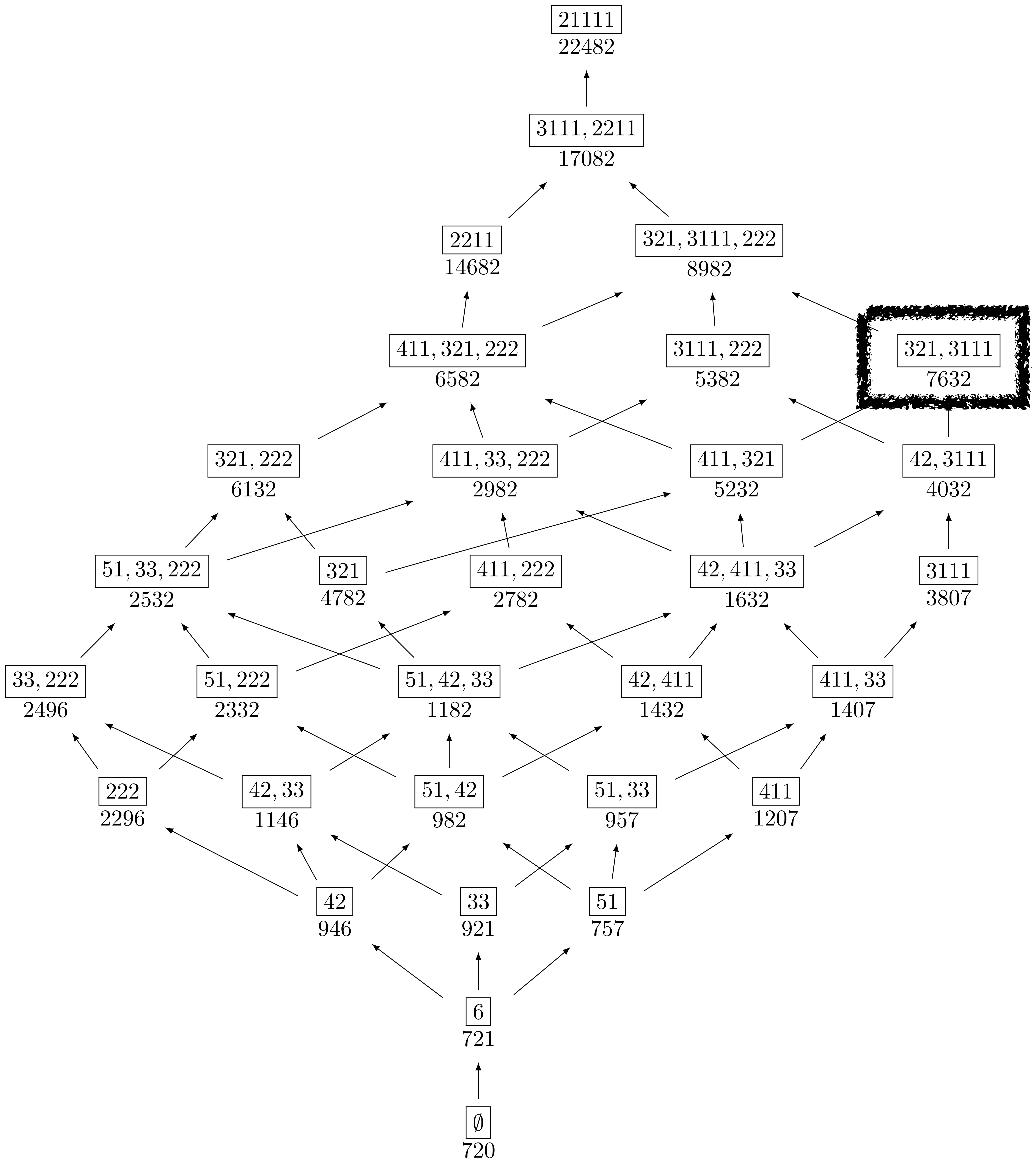
$$\mathrm{Res}^{U_6}_{U_6^I} W^{(\square,\square,\square)}_{U_6} \cong W^{(\square,\square,\square)}_{U_6^I}$$



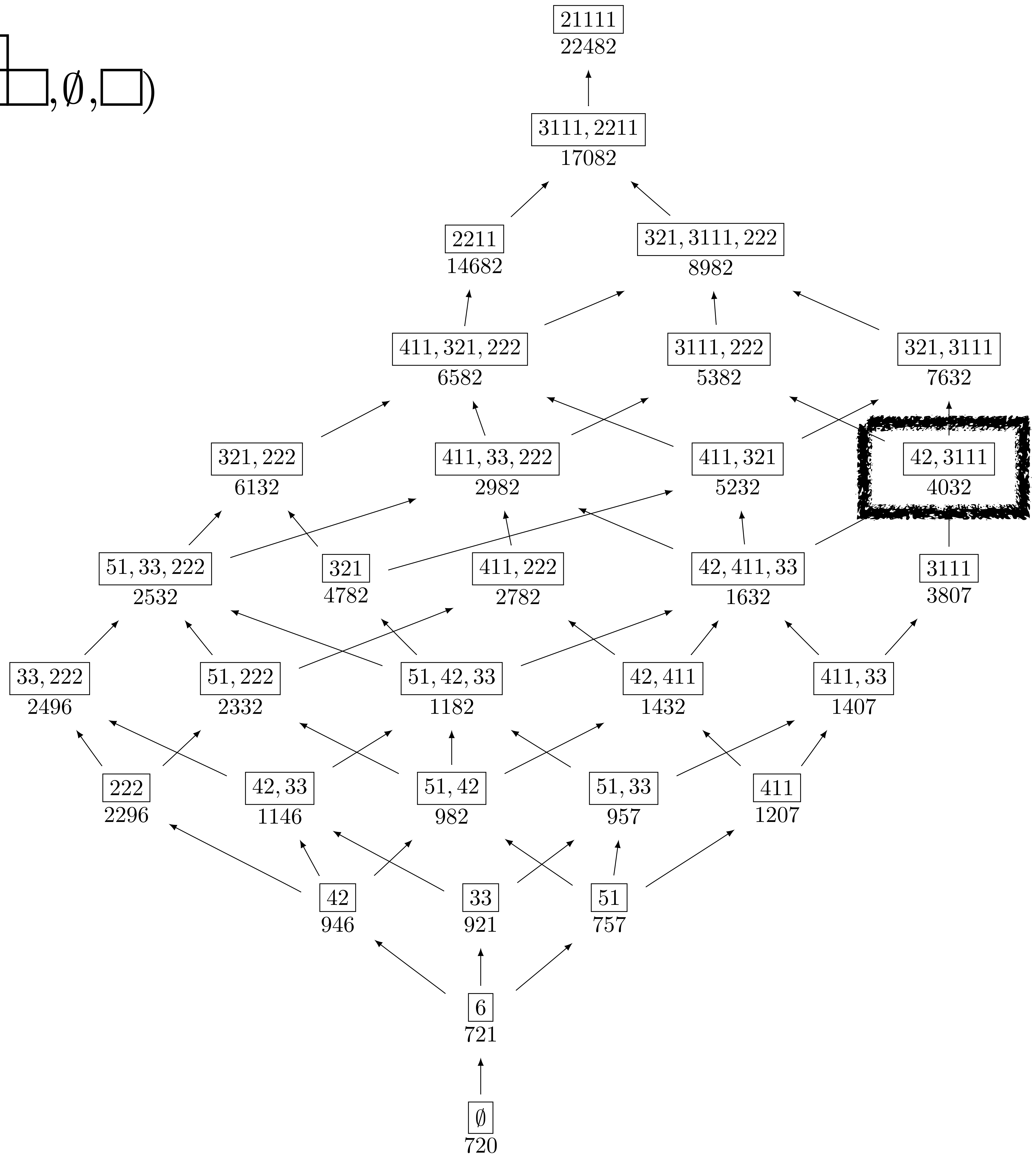
$$\mathrm{Res}^{U_6}_{U_6^I} W^{(\square,\square,\square)}_{U_6} \cong W^{(\square,\square,\square)}_{U_6^I}$$



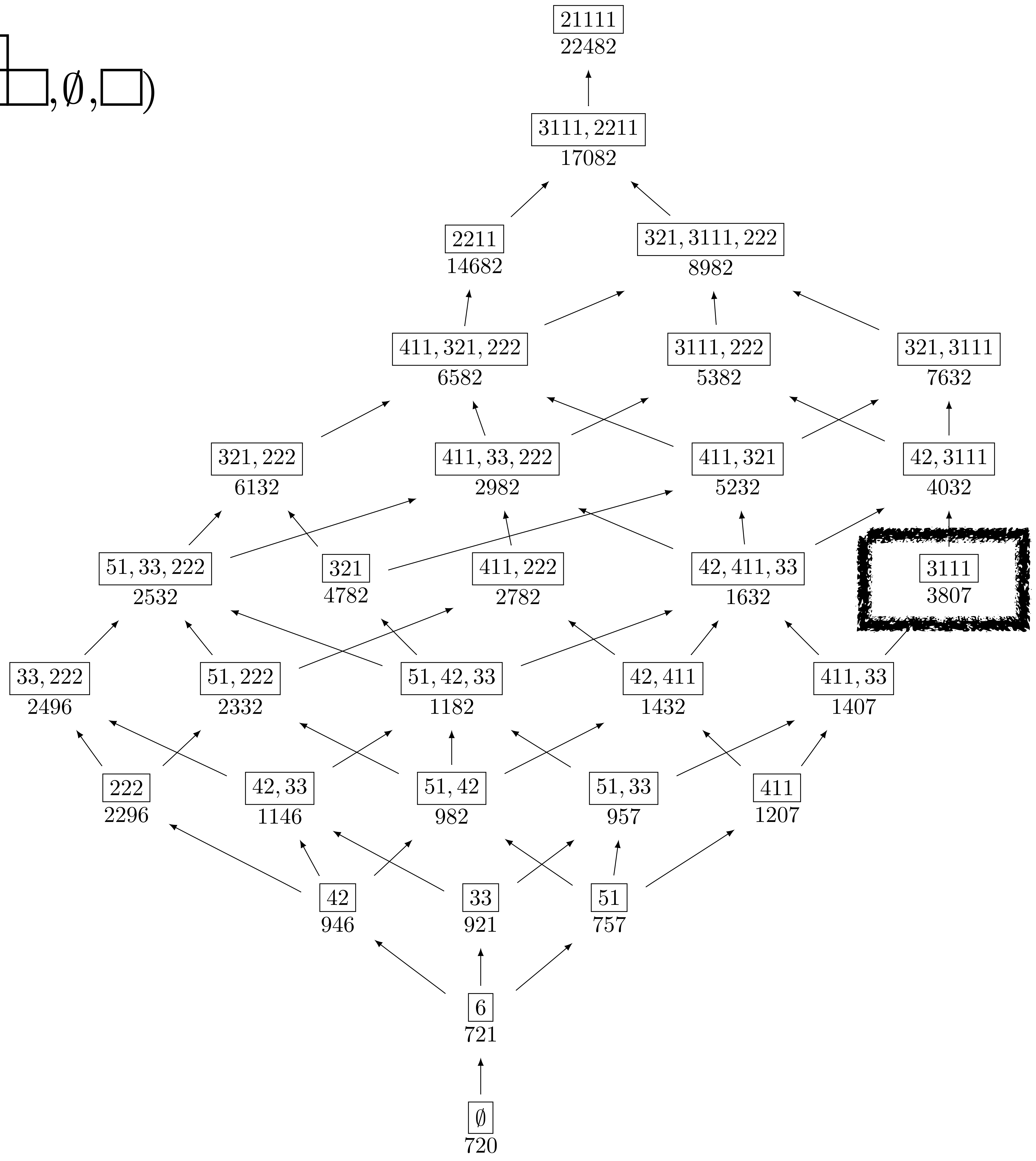
$$\mathrm{Res}^{U_6}_{U_6^I} W^{(\square,\square,\square)}_{U_6} \cong W^{(\square,\square,\square)}_{U_6^I}$$



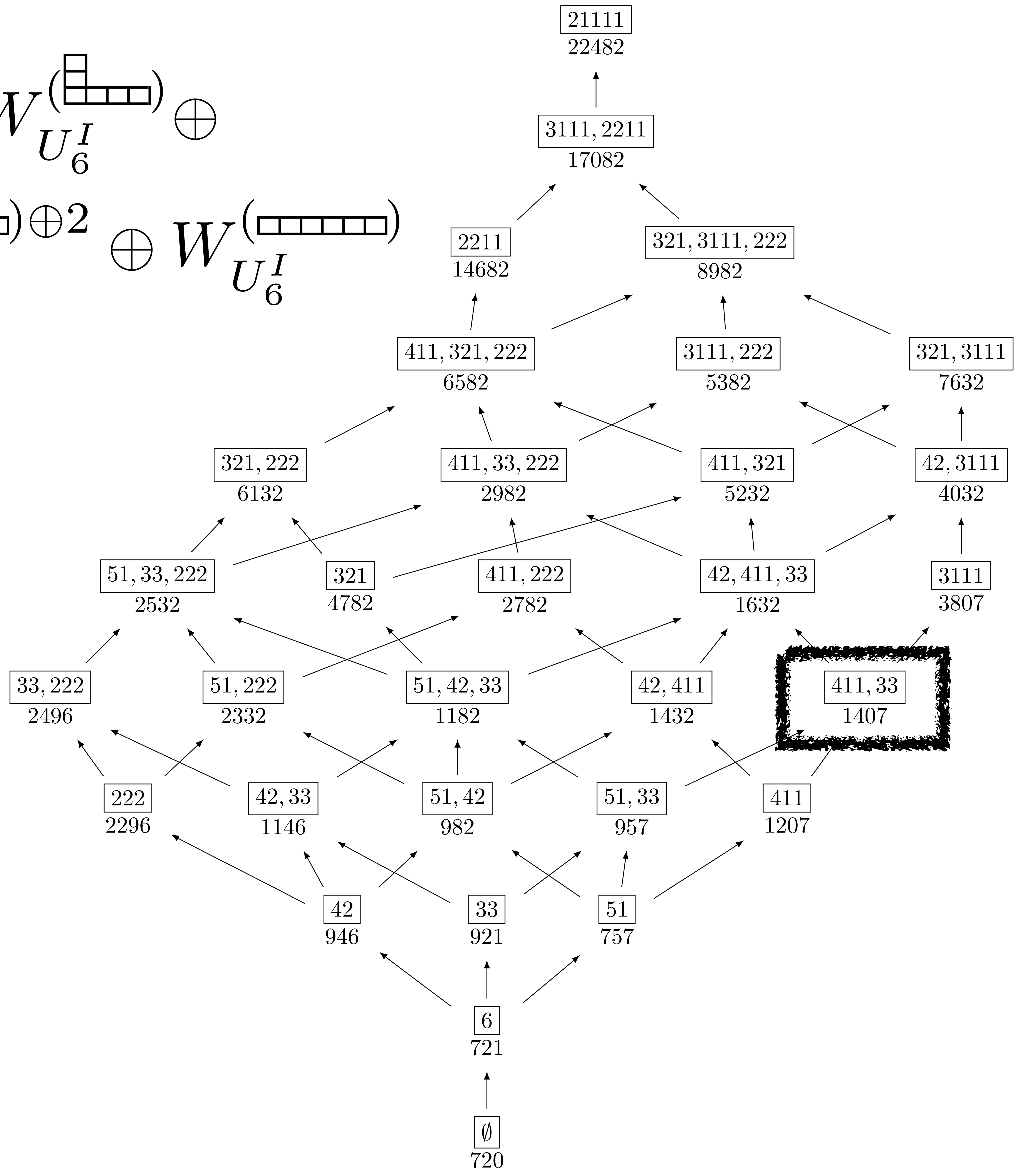
$$\mathrm{Res}^{U_6}_{U_6^I} W_{U_6}^{(\square,\square,\square)} \cong W_{U_6^I}^{(\square\square\square,\emptyset,\square)} \oplus W_{U_6^I}^{(\substack{\square\\ \square\square},\emptyset,\square)}$$



$$\operatorname{Res}^{U_6}_{U_6^I} W_{U_6}^{(\square,\square,\square)} \cong W_{U_6^I}^{(\square\square\square,\emptyset,\square)} \oplus W_{U_6^I}^{\left(\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix},\emptyset,\square\right)}$$

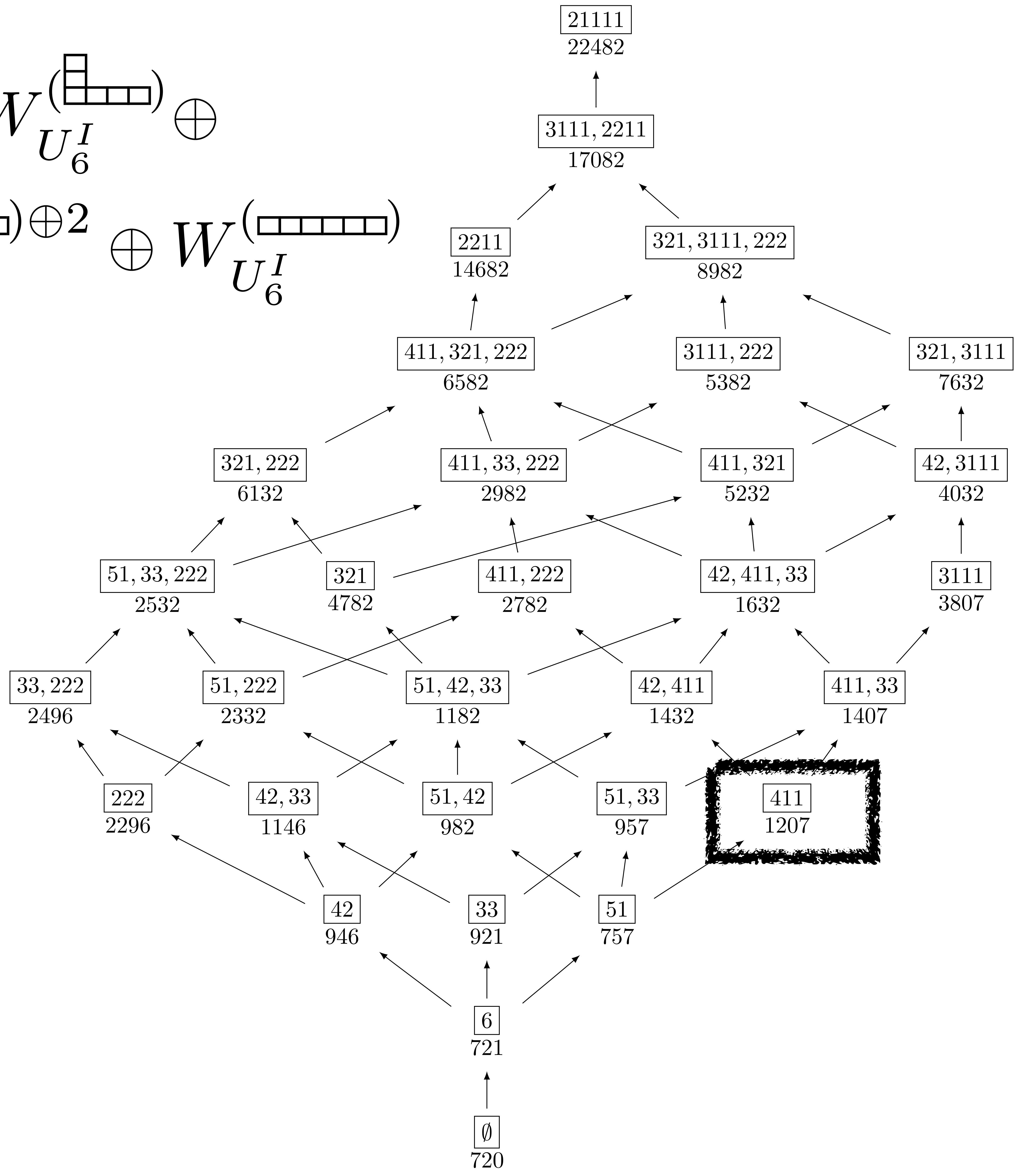


$$\mathrm{Res}^{U_6}_{U_6^I} W_{U_6}^{(\yng(1),\yng(1),\yng(1))} \cong W_{U_6^I}^{(\yng(1,2,3))} \oplus W_{U_6^I}^{(\yng(3,3))} \oplus W_{U_6^I}^{(\yng(1,1,4))} \oplus W_{U_6^I}^{(\yng(2,4))\oplus 2} \oplus W_{U_6^I}^{(\yng(1,5))\oplus 2} \oplus W_{U_6^I}^{(\yng(6))}$$



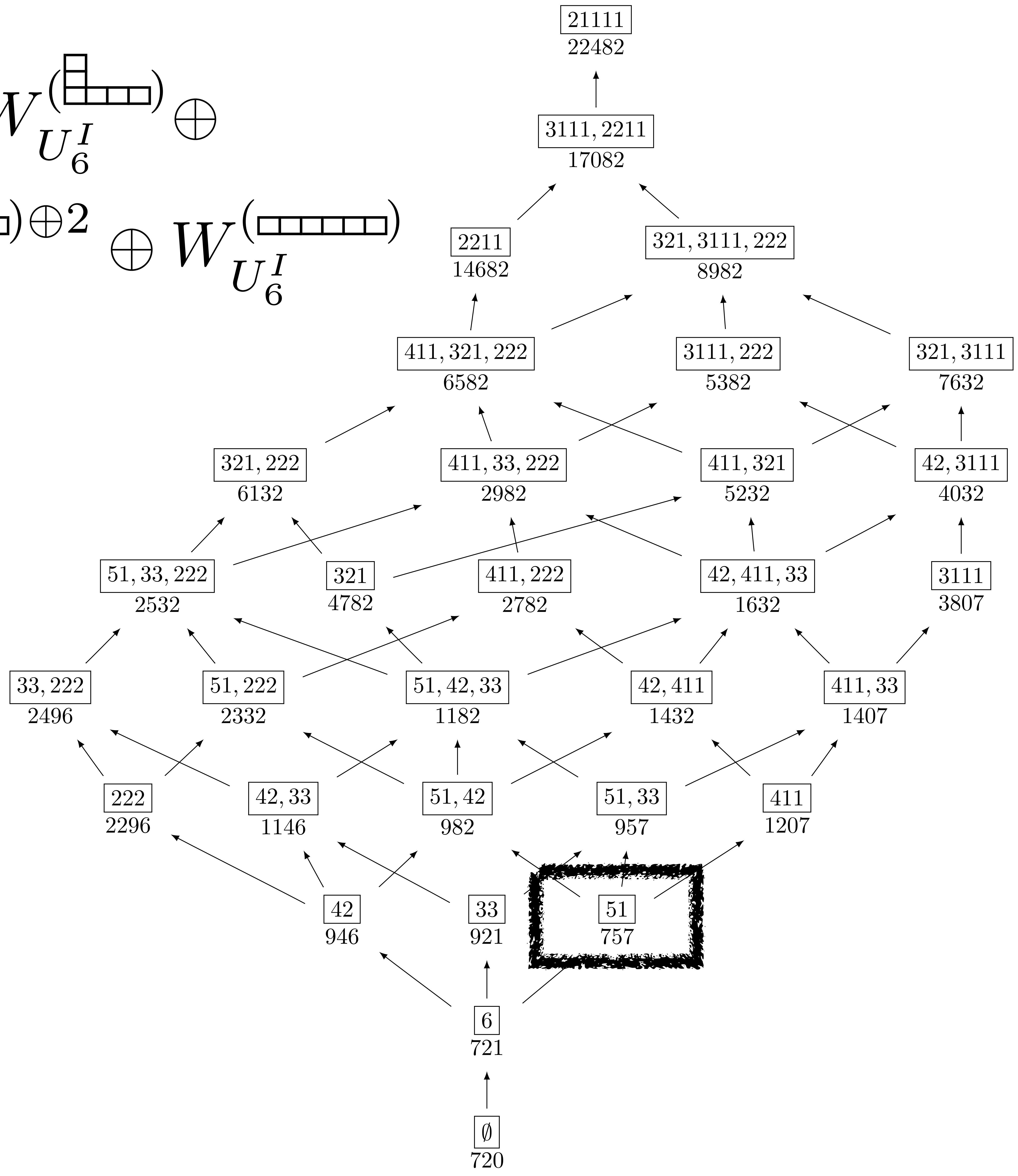
$$\mathrm{Res}^{U_6}_{U_6^I} W^{(\yng(1),\yng(1),\yng(1))}_{U_6} \cong W^{(\yng(3,2,1))}_{U_6^I} \oplus W^{(\yng(3,3))}_{U_6^I} \oplus W^{(\yng(4,1,1))}_{U_6^I} \oplus$$

$$W^{(\yng(4,2))}_{U_6^I}{}^{\oplus 2} \oplus W^{(\yng(5,1))}_{U_6^I}{}^{\oplus 2} \oplus W^{(\yng(6))}_{U_6^I}$$

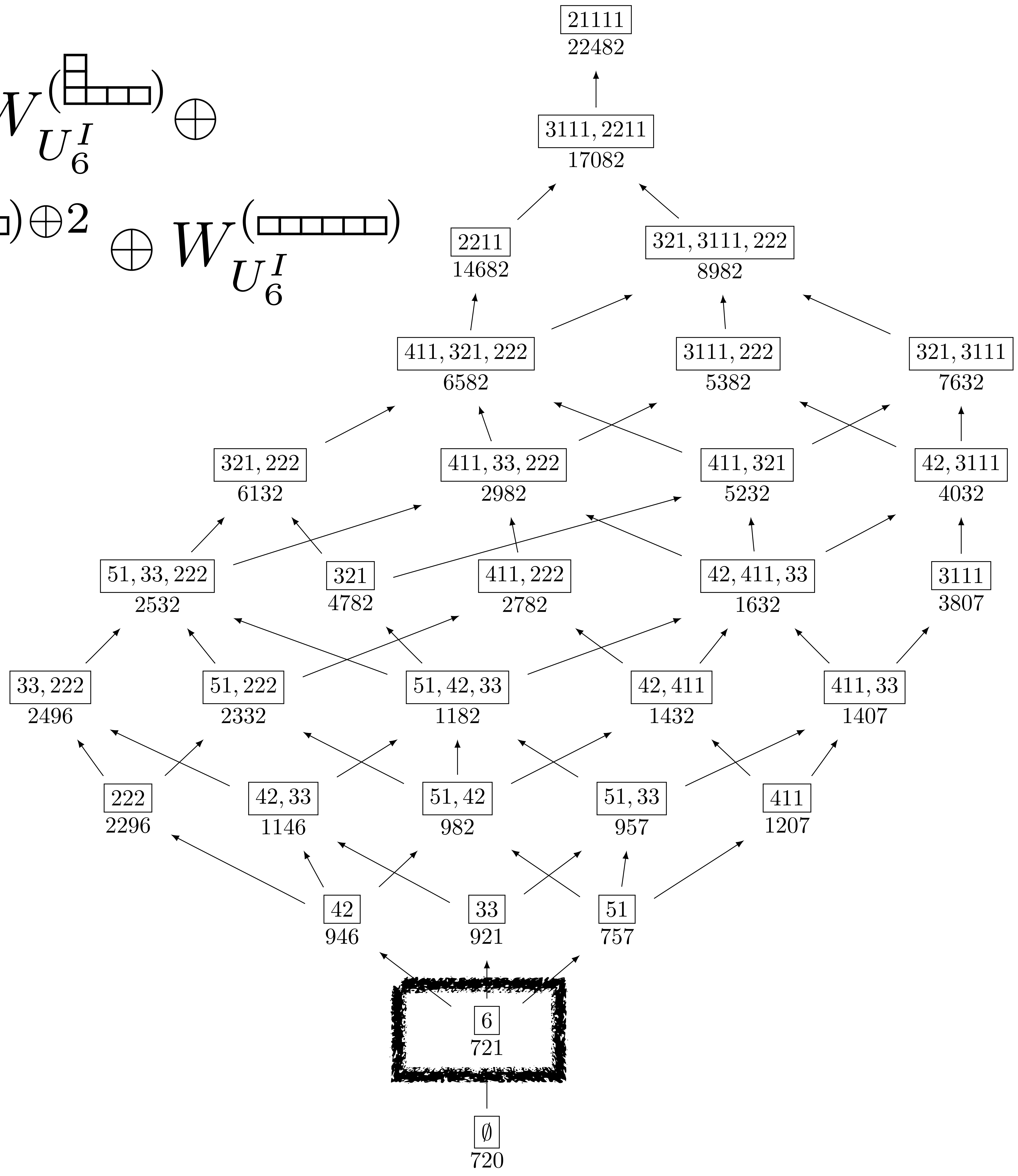


$$\mathrm{Res}^{U_6}_{U_6^I} W_{U_6}^{(\ydiagram{1},\ydiagram{1},\ydiagram{1})} \cong W_{U_6^I}^{(\ydiagram{1,2,3})} \oplus W_{U_6^I}^{(\ydiagram{3,3})} \oplus W_{U_6^I}^{(\ydiagram{1,1,4})} \oplus$$

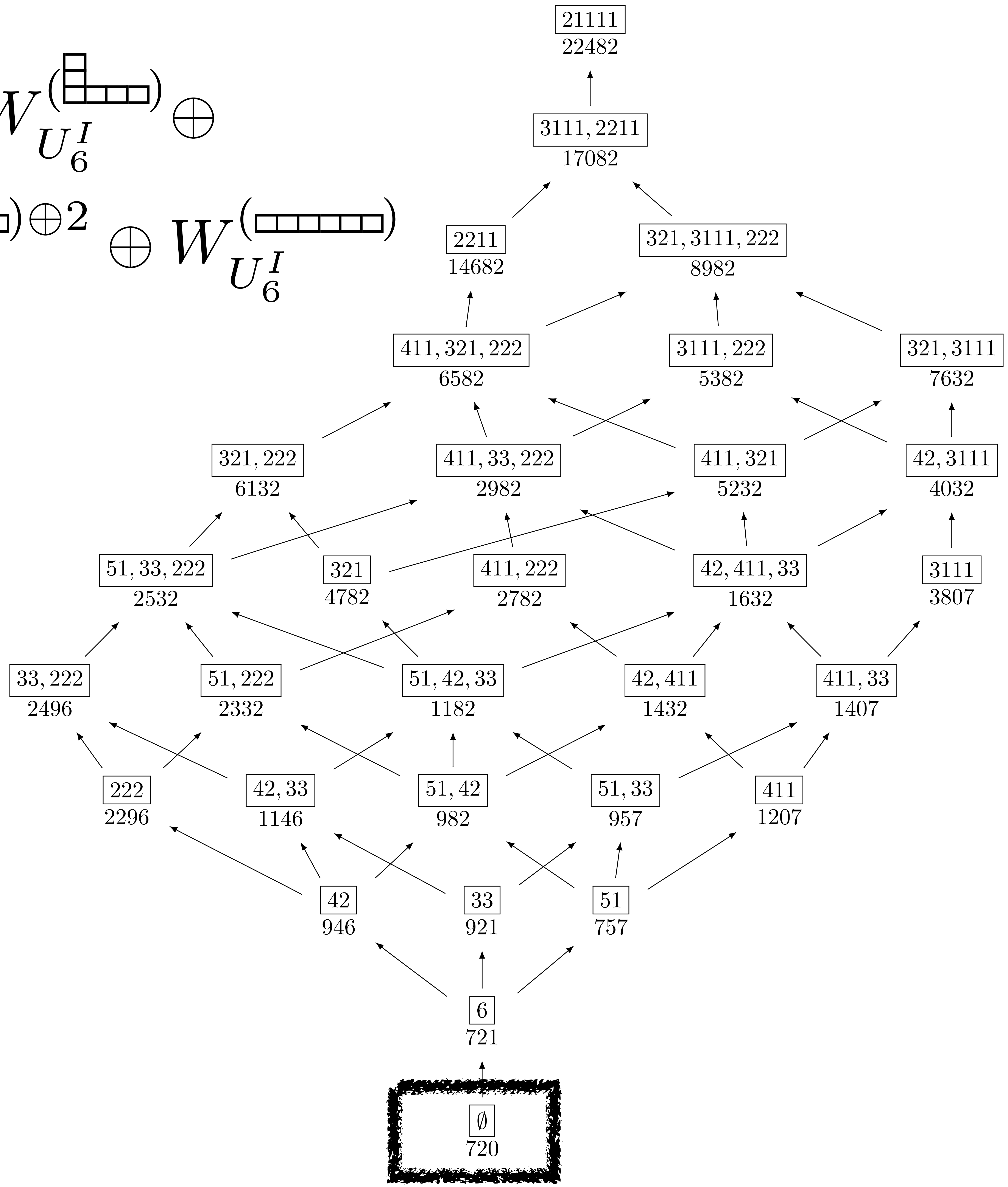
$$W_{U_6^I}^{(\ydiagram{2,4})\oplus 2} \oplus W_{U_6^I}^{(\ydiagram{1,5})\oplus 2} \oplus W_{U_6^I}^{(\ydiagram{6})}$$



$$\mathrm{Res}^{U_6}_{U_6^I} W_{U_6}^{(\ydiagram{1},\ydiagram{1},\ydiagram{1})} \cong W_{U_6^I}^{(\ydiagram{3,2,1})} \oplus W_{U_6^I}^{(\ydiagram{3,3})} \oplus W_{U_6^I}^{(\ydiagram{4,1,1})} \oplus W_{U_6^I}^{(\ydiagram{4,2})\oplus 2} \oplus W_{U_6^I}^{(\ydiagram{5,1})\oplus 2} \oplus W_{U_6^I}^{(\ydiagram{6})}$$



$$\operatorname{Res}^{U_6}_{U_6^I} W_{U_6}^{(\ydiagram{1},\ydiagram{1},\ydiagram{1})} \cong W_{U_6^I}^{(\ydiagram{3,2,1})} \oplus W_{U_6^I}^{(\ydiagram{3,3})} \oplus W_{U_6^I}^{(\ydiagram{4,1,1})} \oplus W_{U_6^I}^{(\ydiagram{4,2})\oplus 2} \oplus W_{U_6^I}^{(\ydiagram{5,1})\oplus 2} \oplus W_{U_6^I}^{(\ydiagram{6})}$$



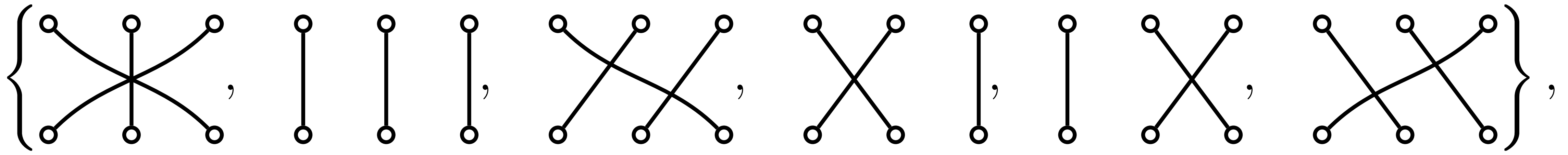
Structure of uniform block partition monoid

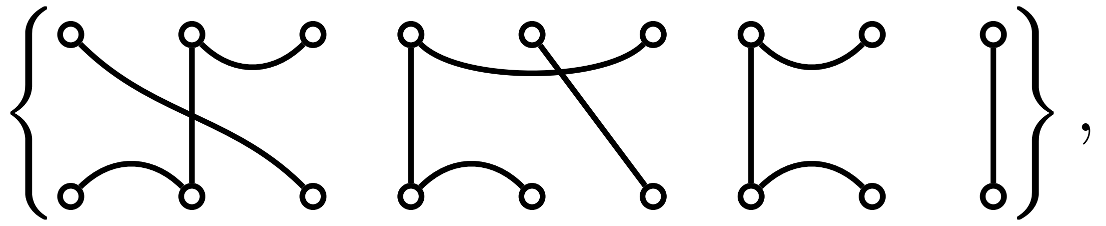
- the uniform block permutation (UBP) algebra is a monoid algebra
- monoid theory $\Longrightarrow$ UBP_k is a union of $\mathcal{J}$-classes & $\mathcal{L}$-classes

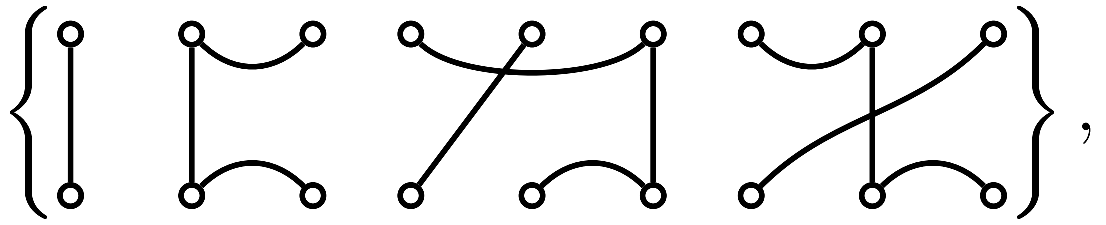
$x \equiv_{\mathcal{J}} y \Longleftrightarrow MxM = MyM$ $\quad$ $x \equiv_{\mathcal{L}} y \Longleftrightarrow Mx = My$

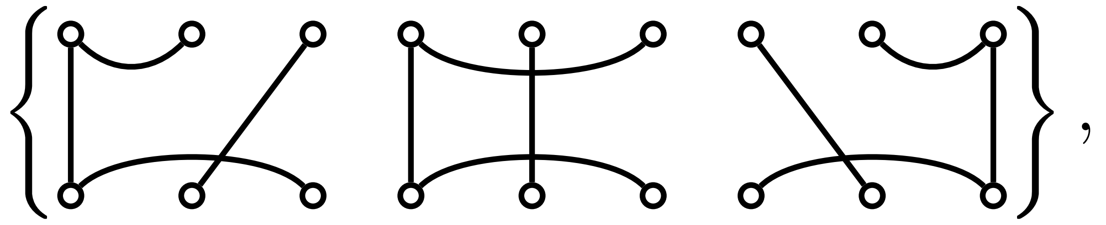
$\mathcal{L}$-classes are indexed by set partitions of k

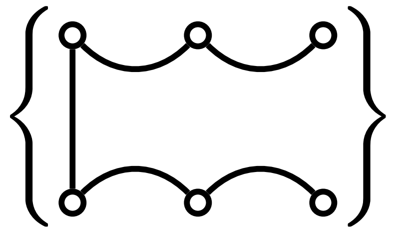
L_π = bottom row of diagram is equal to π

$L_{1|2|3} =$


$L_{12|3} =$


$L_{1|23} =$


$L_{13|2} =$


$L_{123} =$


$$\mathrm{UBP}_k = \bigcup_{\pi \vdash [k]} L_\pi$$