k-Schur functions indexed by a maximal rectangle

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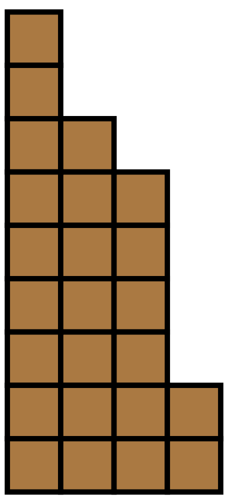
joint work with Chris Berg, Nantel Bergeron, Hugh Thomas

Lapointe-Morse (2005)

$$\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k]$$

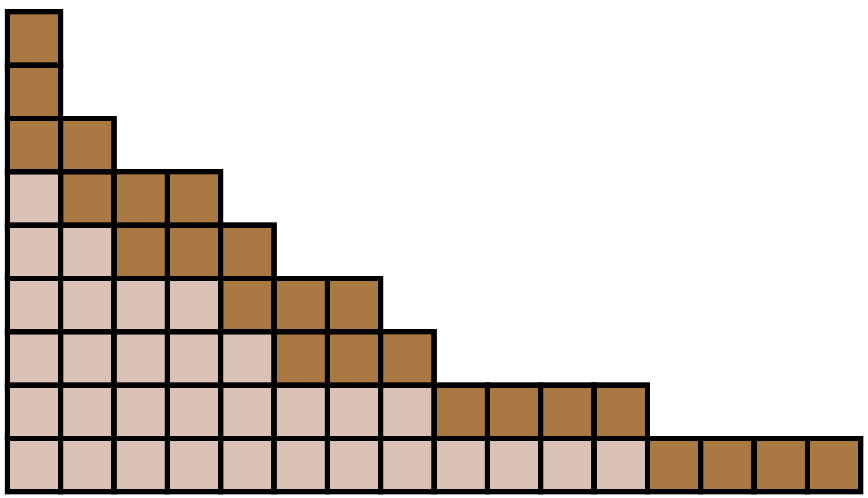
definition of a basis $\{s_\lambda^{(k)}\}_\lambda$ indexed by partitions

λ partitions $\lambda_1 \leq k$



$\lambda = \mathfrak{p}(\gamma)$

(k+1)-cores=partitions with no (k+1)-hooks



$\mathfrak{c}(\lambda) = \gamma$

Example: $k = 4$

Lapointe-Morse defintion of k-Schur functions

$\{s_\lambda^{(k)}\}_\lambda$ basis of algebra $\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k]$

satisfying

$$h_r s_\lambda^{(k)} = \sum_\mu s_\mu^{(k)}$$

$\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ is a horizontal strip, $\lambda \leq \mu$

$|\mu| = |\lambda| + r$

This is a recursive definition because of triangularity considerations

Example: k=3 to calculate $s^{(3)}_{(2,2,1)}$

the 3-Pieri rule says:

$$h_2 s^{(3)}_{(2,1)} = s^{(3)}_{(2,2,1)} + s^{(3)}_{(3,1,1)}$$

We may assume (inductively) that expansions of $s^{(3)}_{(3,1,1)}$ and $s^{(3)}_{(2,1)}$ are known in terms of the generators

In particular, if hook λ is small (less or equal k) then

$$s^{(k)}_{\lambda} = s_{\lambda}$$

Affine symmetric group W of type $\widetilde{A}_k$

W generated by elements $\{s_0, s_1, s_2, \ldots, s_k\}$

$$s_i^2 = 1$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad i, i+1 \ (mod\ k+1)$$

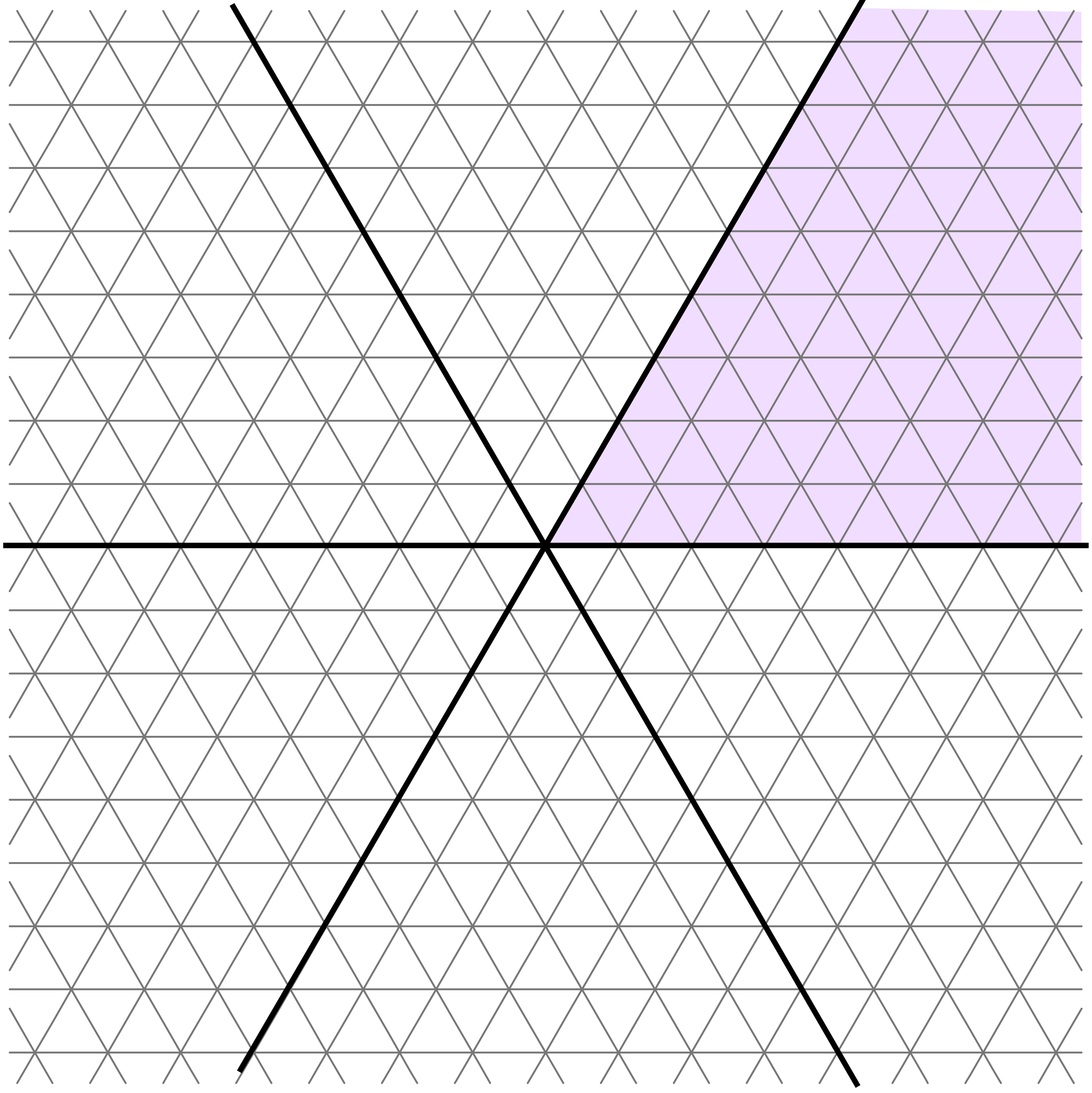
$$s_i s_j = s_j s_i \qquad i - j \not\equiv k, 0, 1 \ (mod\ k+1)$$

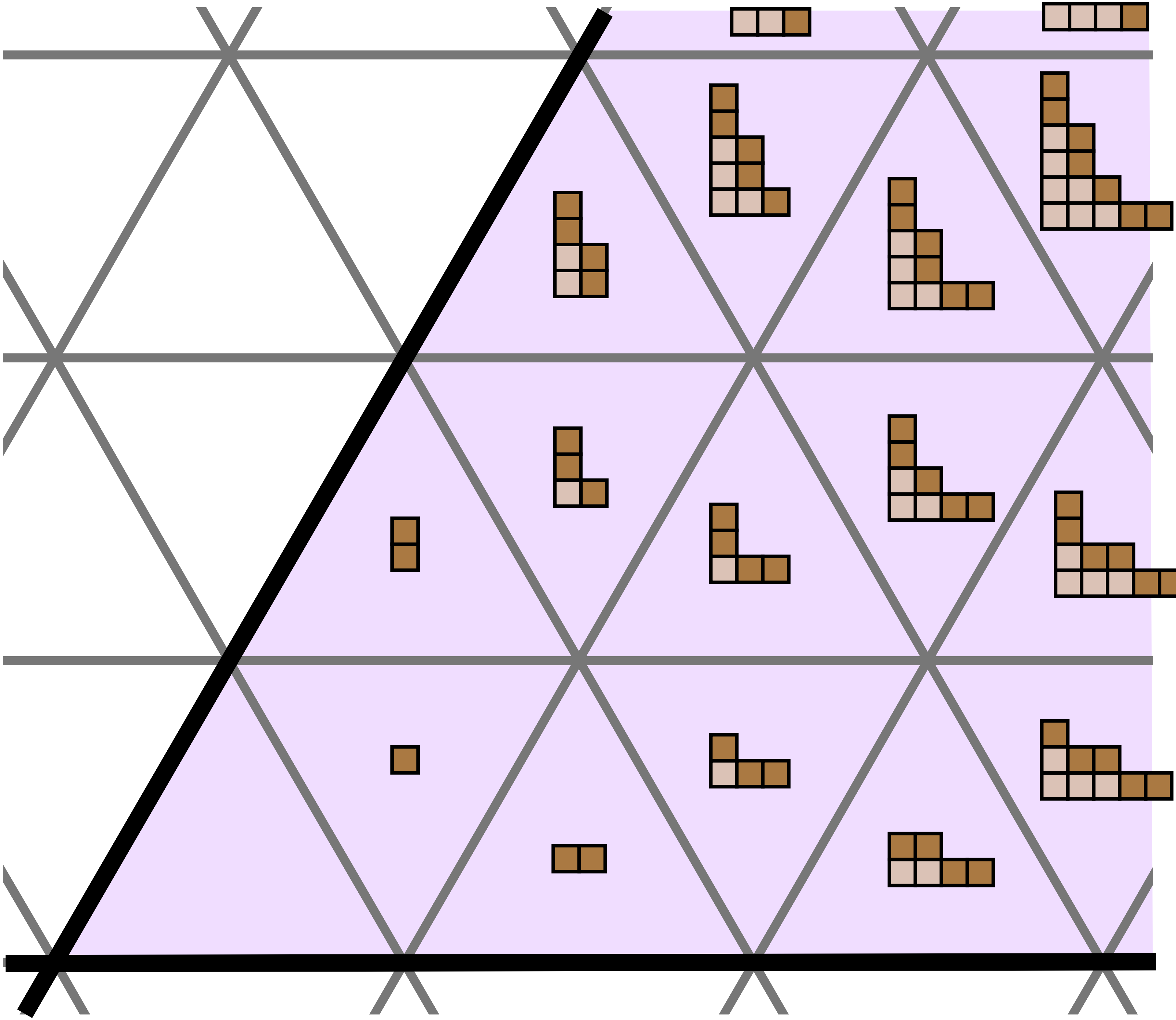
W_0 is the subgroup generated by $\{s_1, s_2, \ldots, s_k\}$

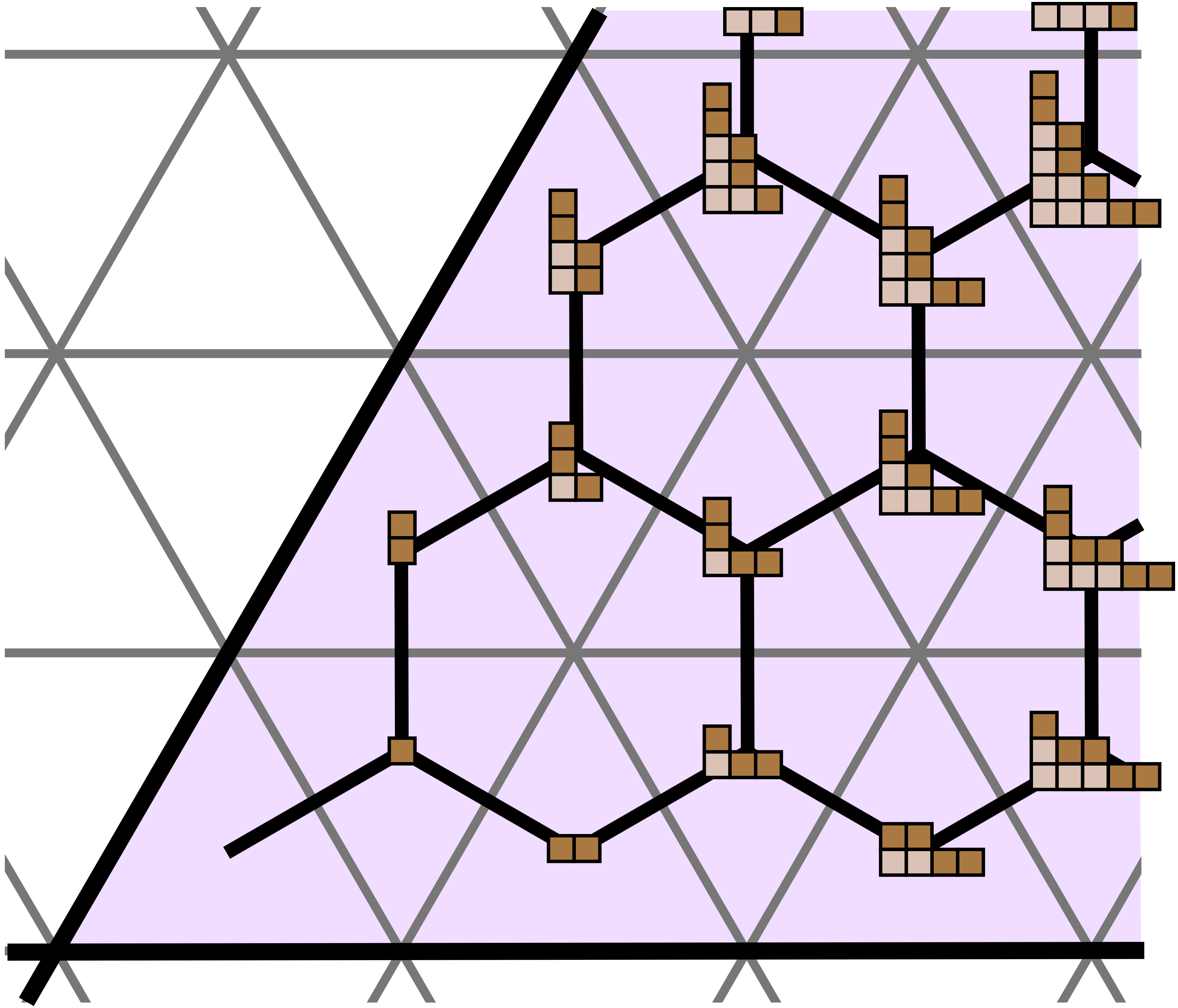
W/W_0 = cosets of W_0 are in bijection with

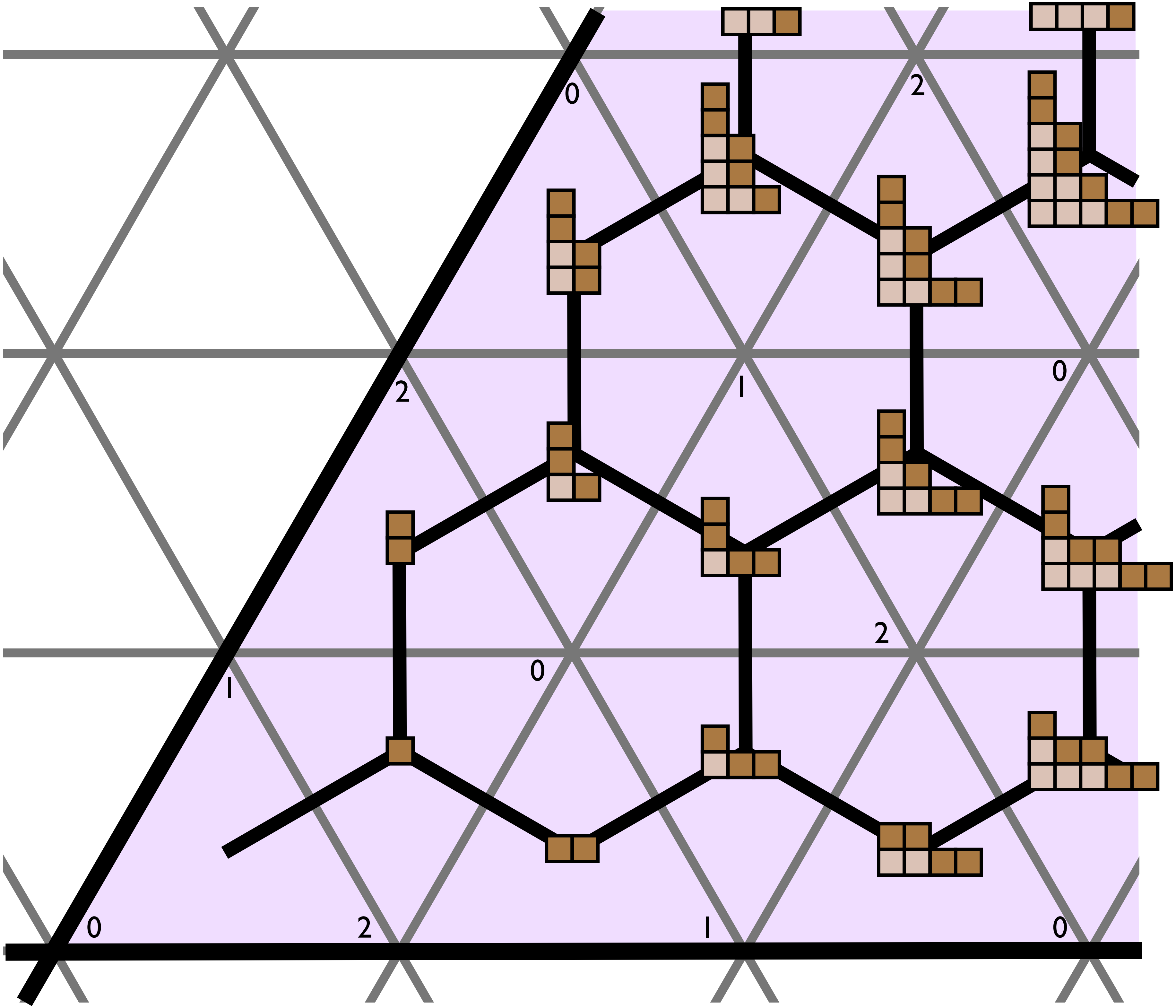
k-bounded partitions/(k+1)-cores

k=2









0
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1
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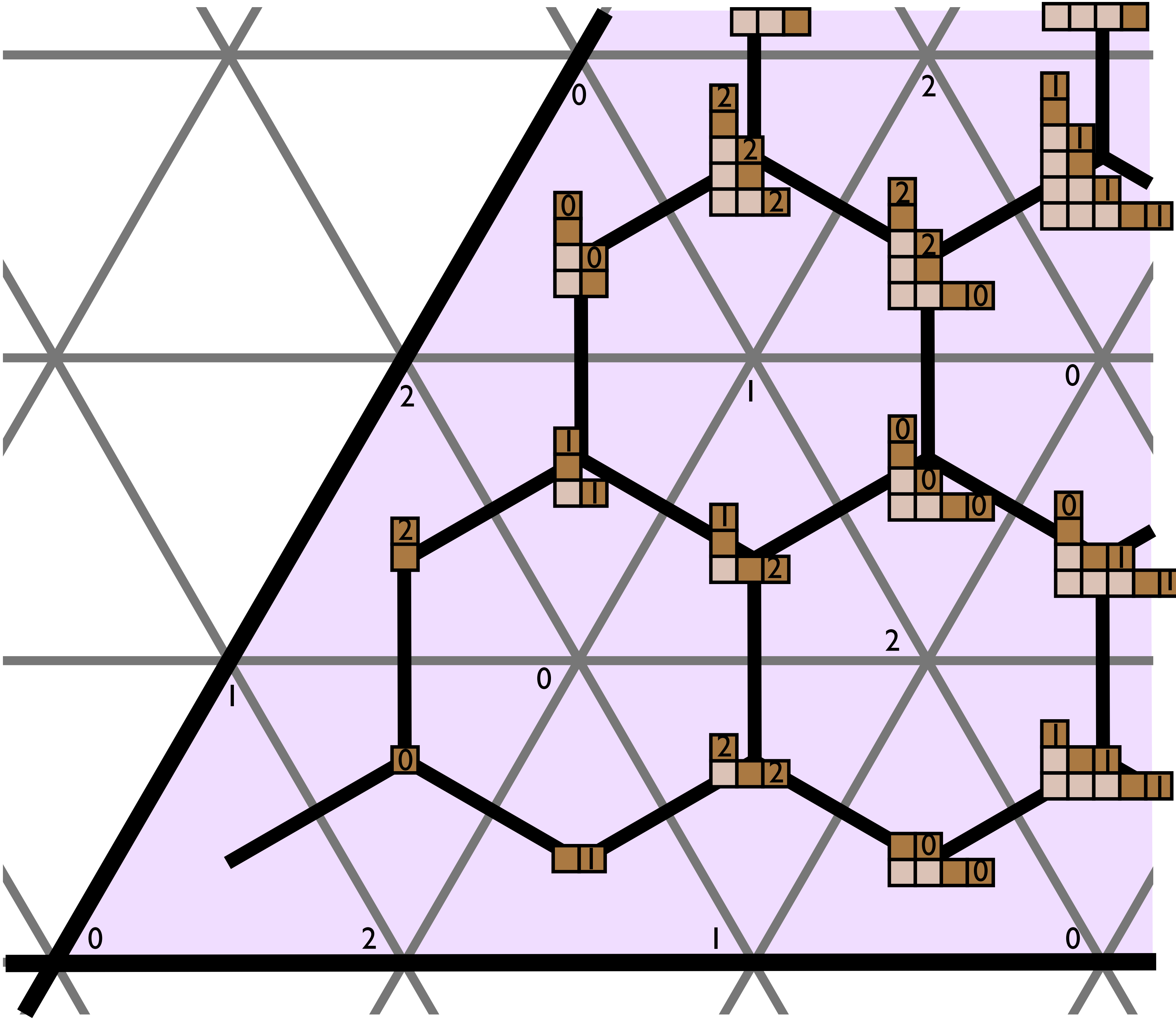
0
0
2
1
0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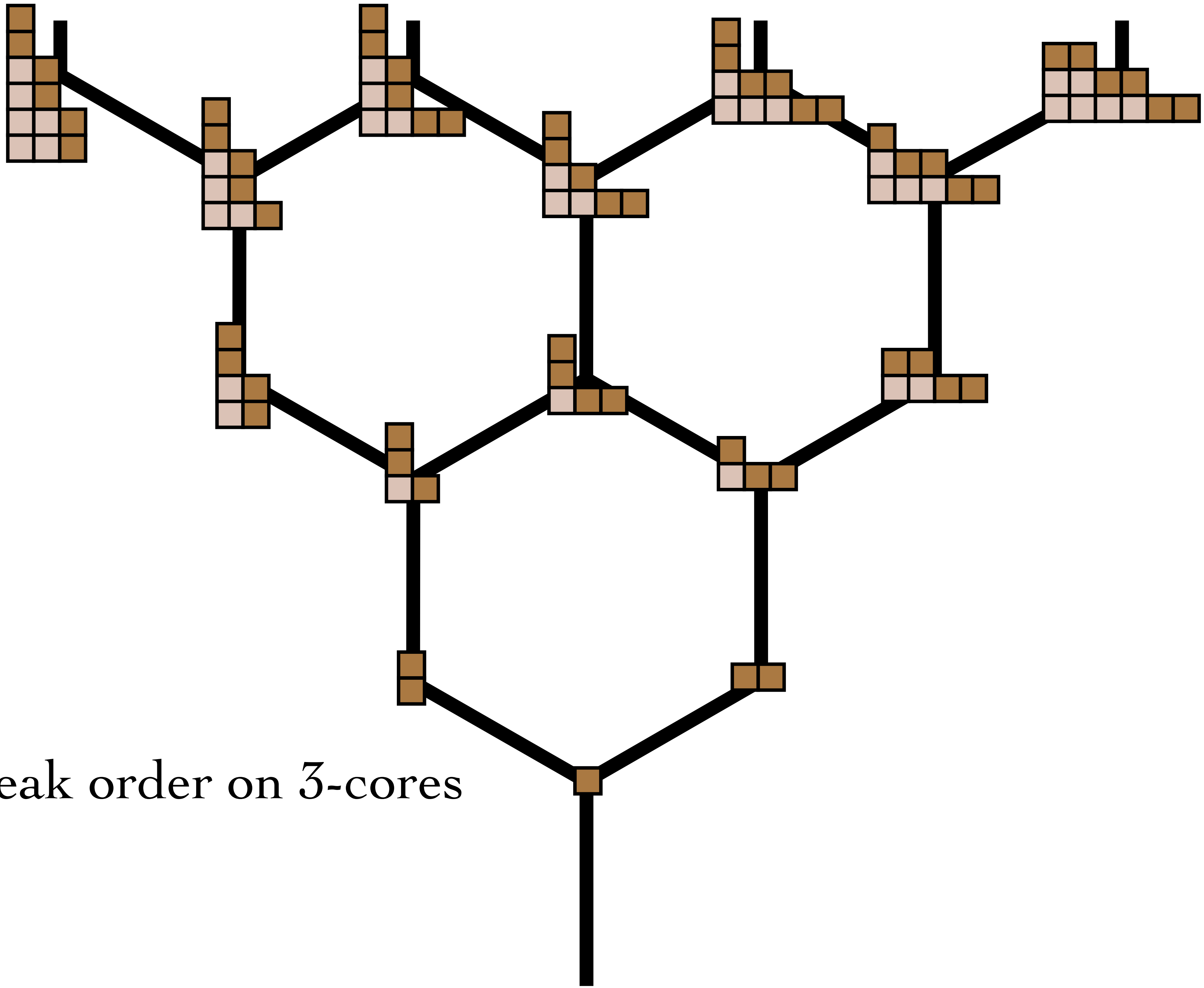
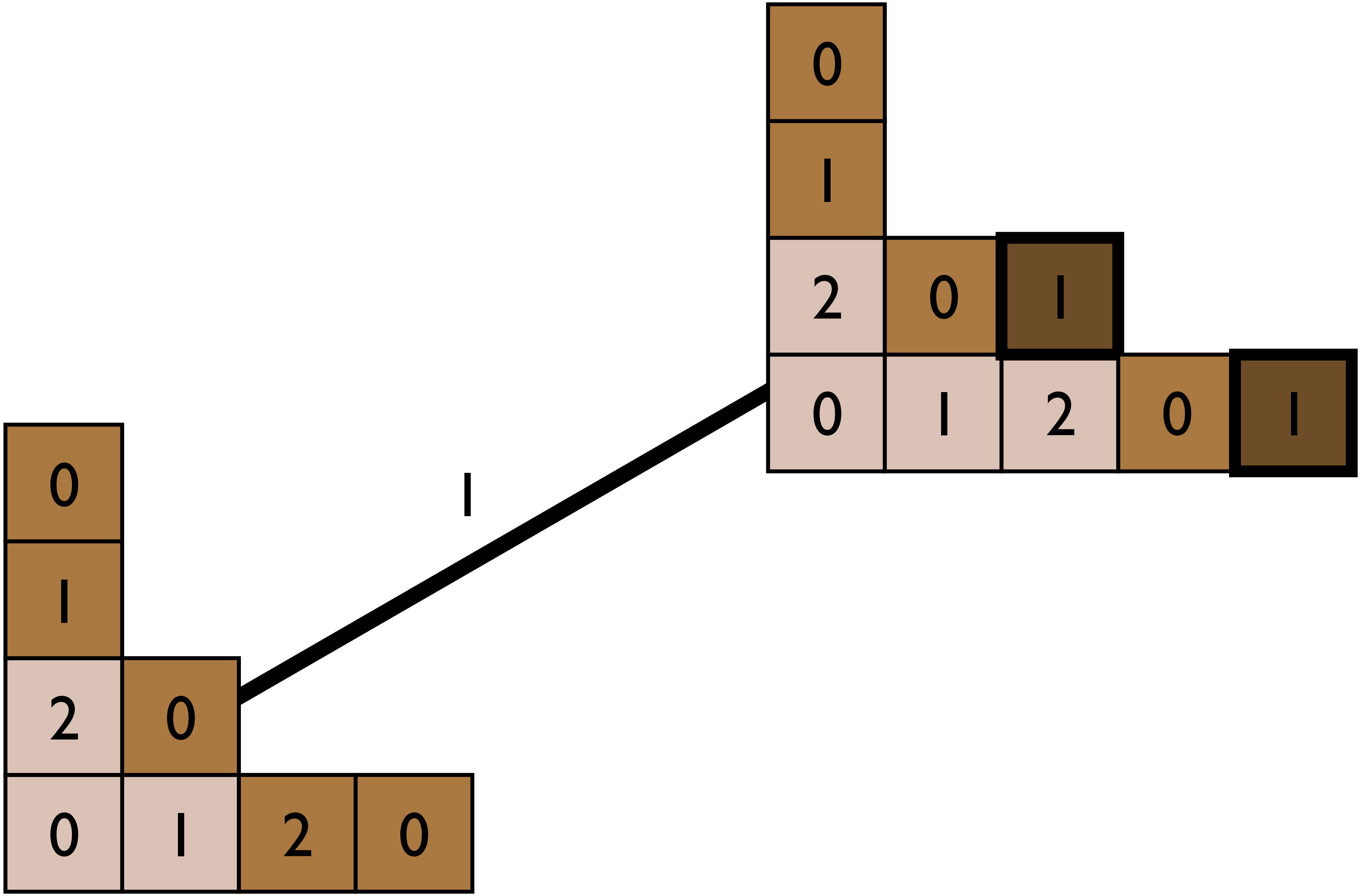


Weak order on 3-cores

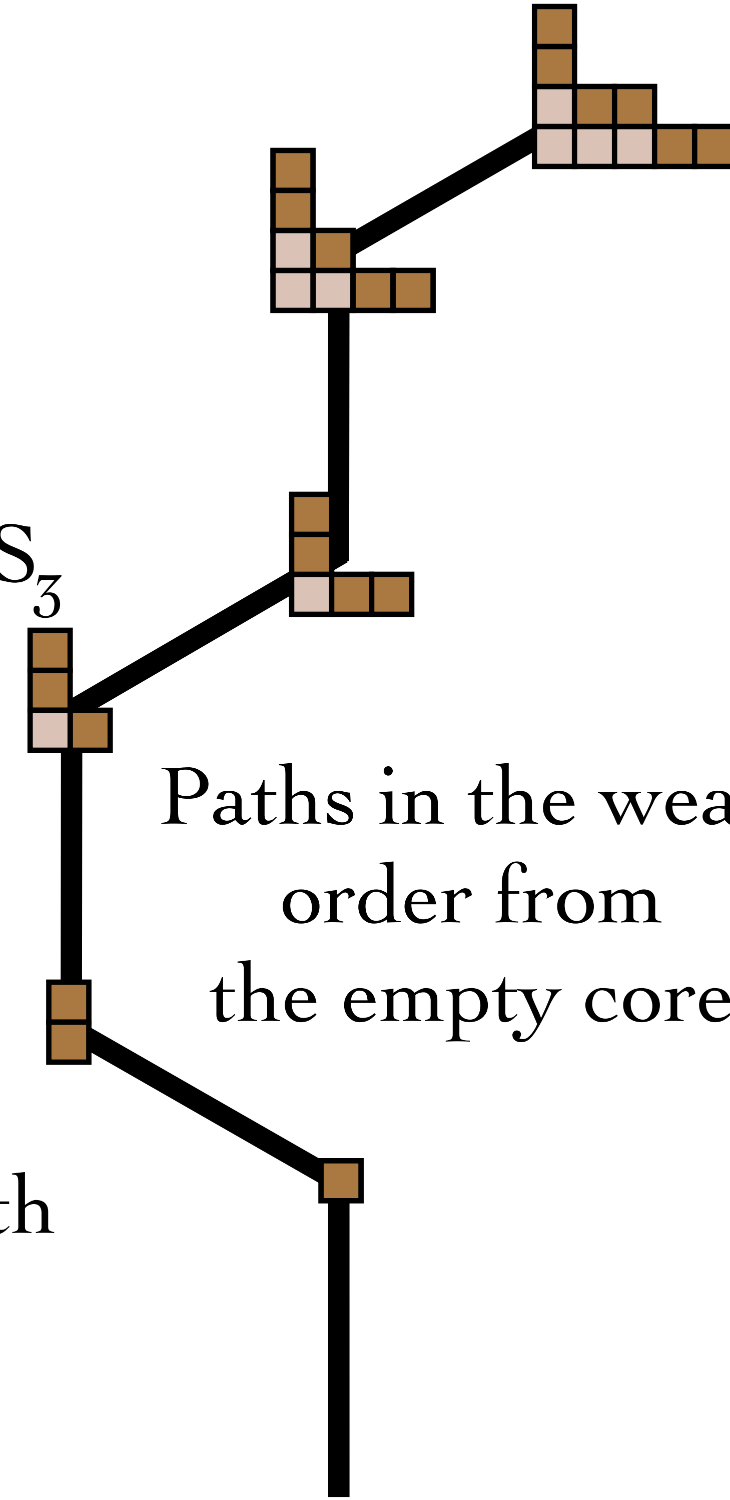


0
1
2 0
0 1 2 0
1
0
1
2 0 1
0 1 2 0 1

3-cores/2-bounded partitions

Cosets of S_3 as subgroup of affine S_3

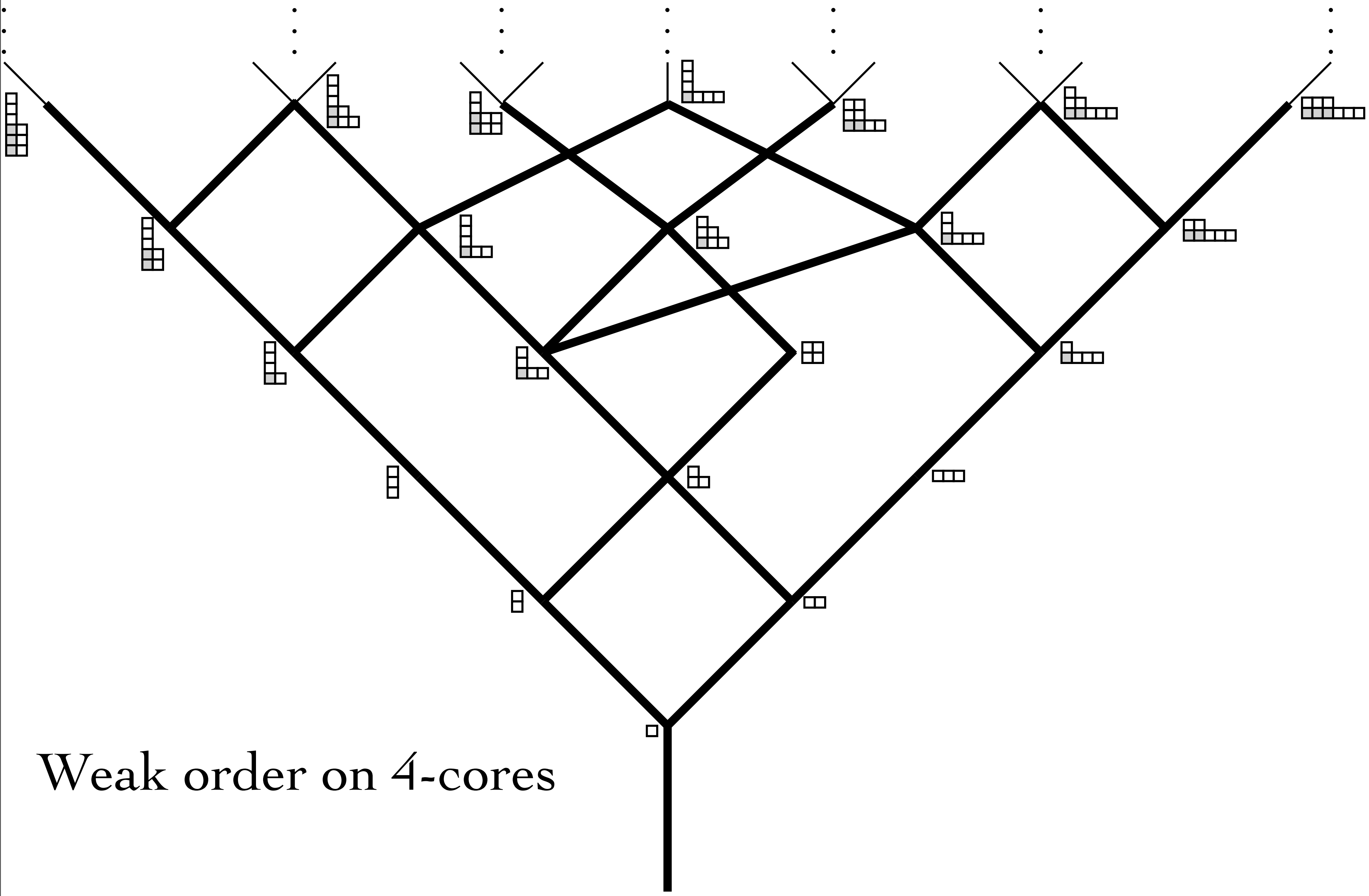
minimal length
coset representatives



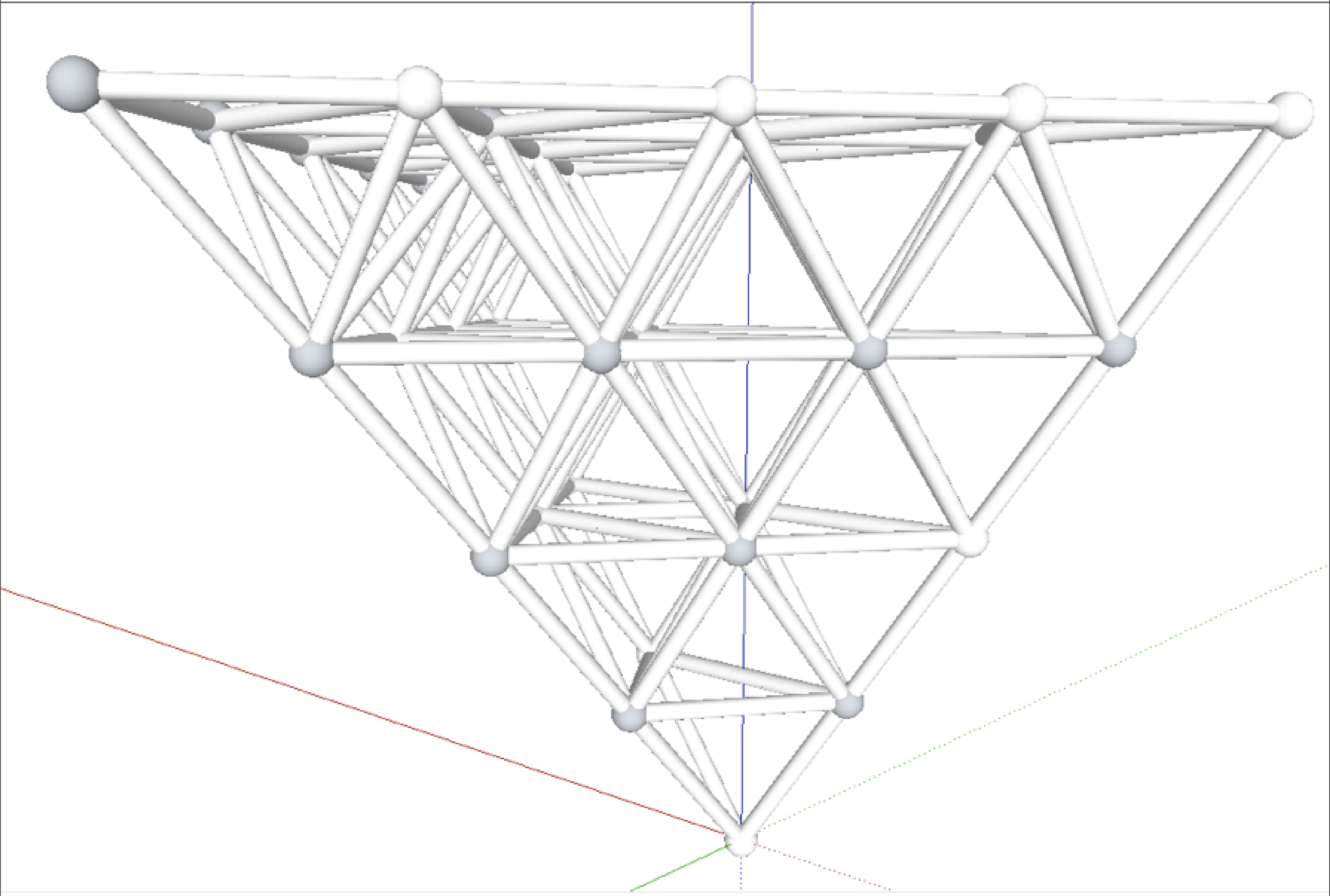
Paths in the weak
order from
the empty core

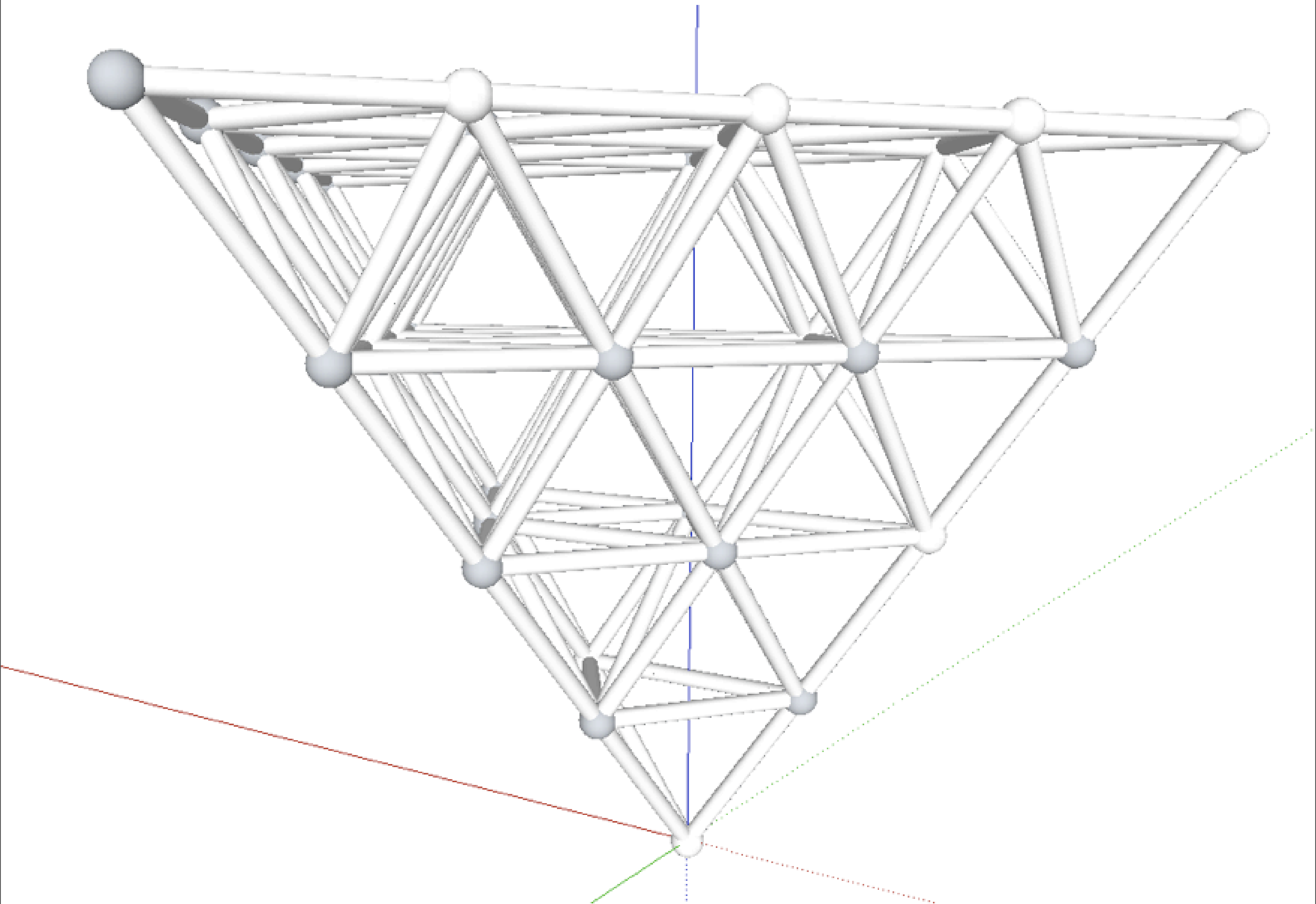
Reduced word for minimal length
coset representatives

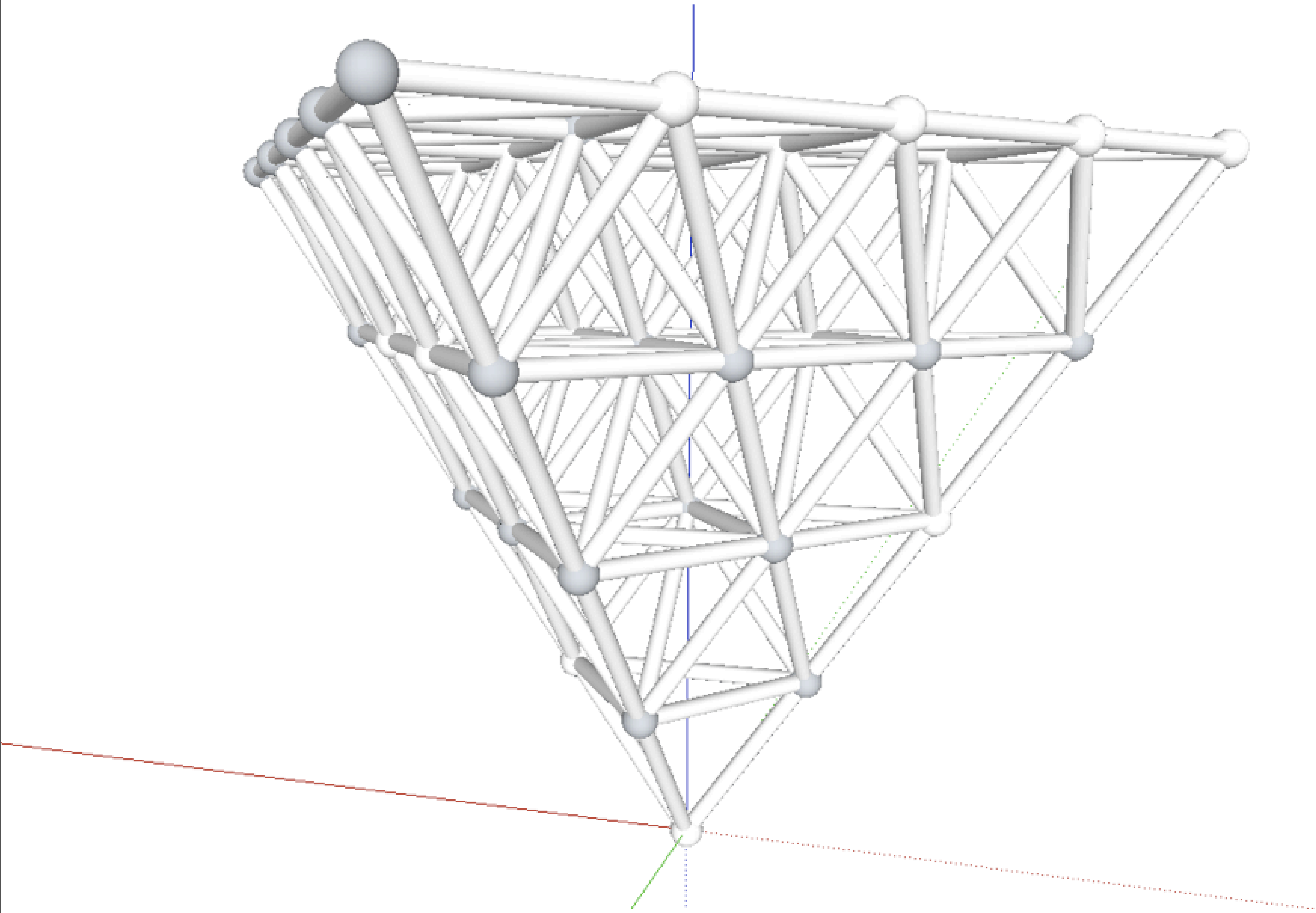
$s_1 s_0 s_2 s_1 s_2 s_0$

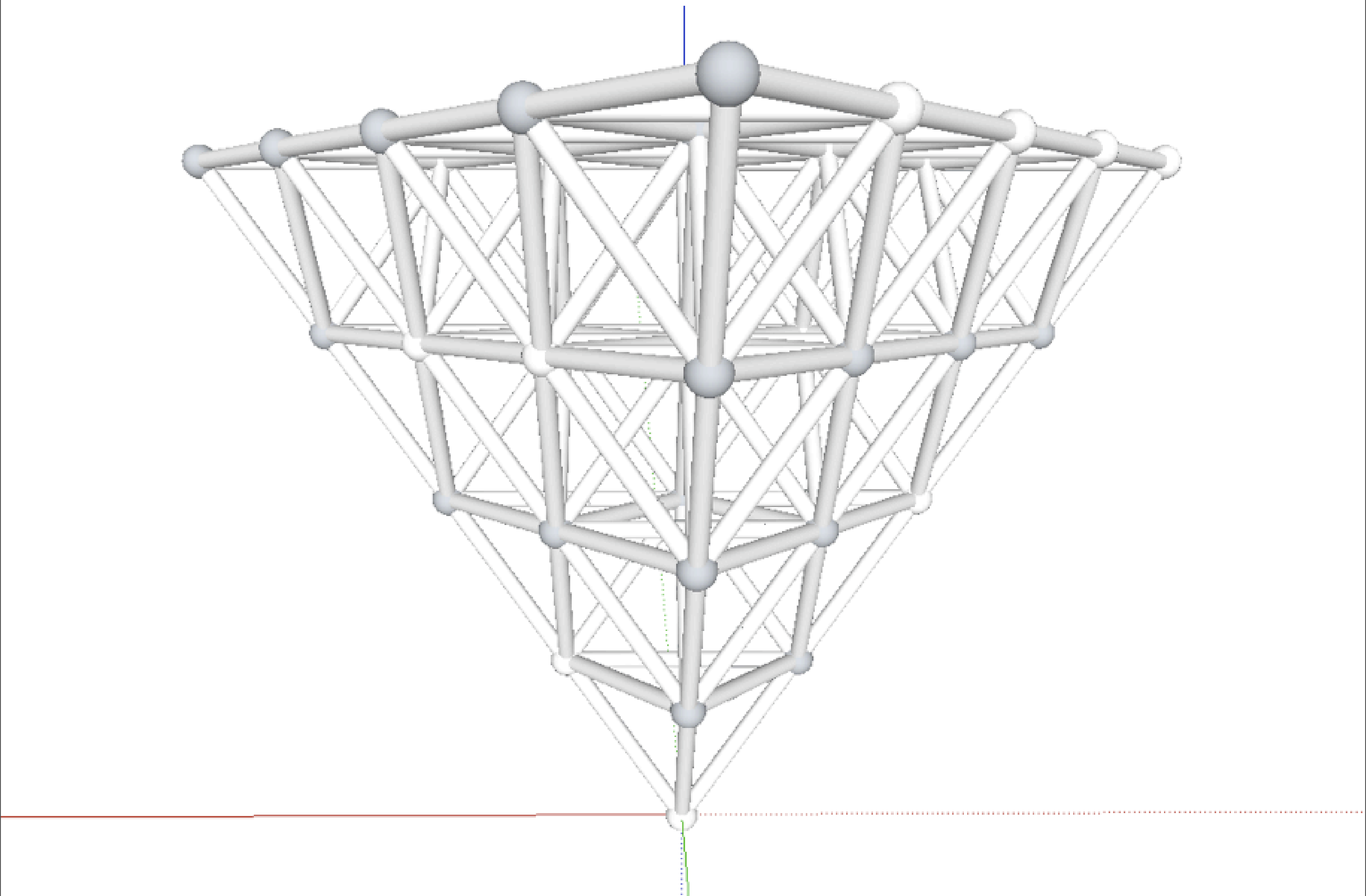


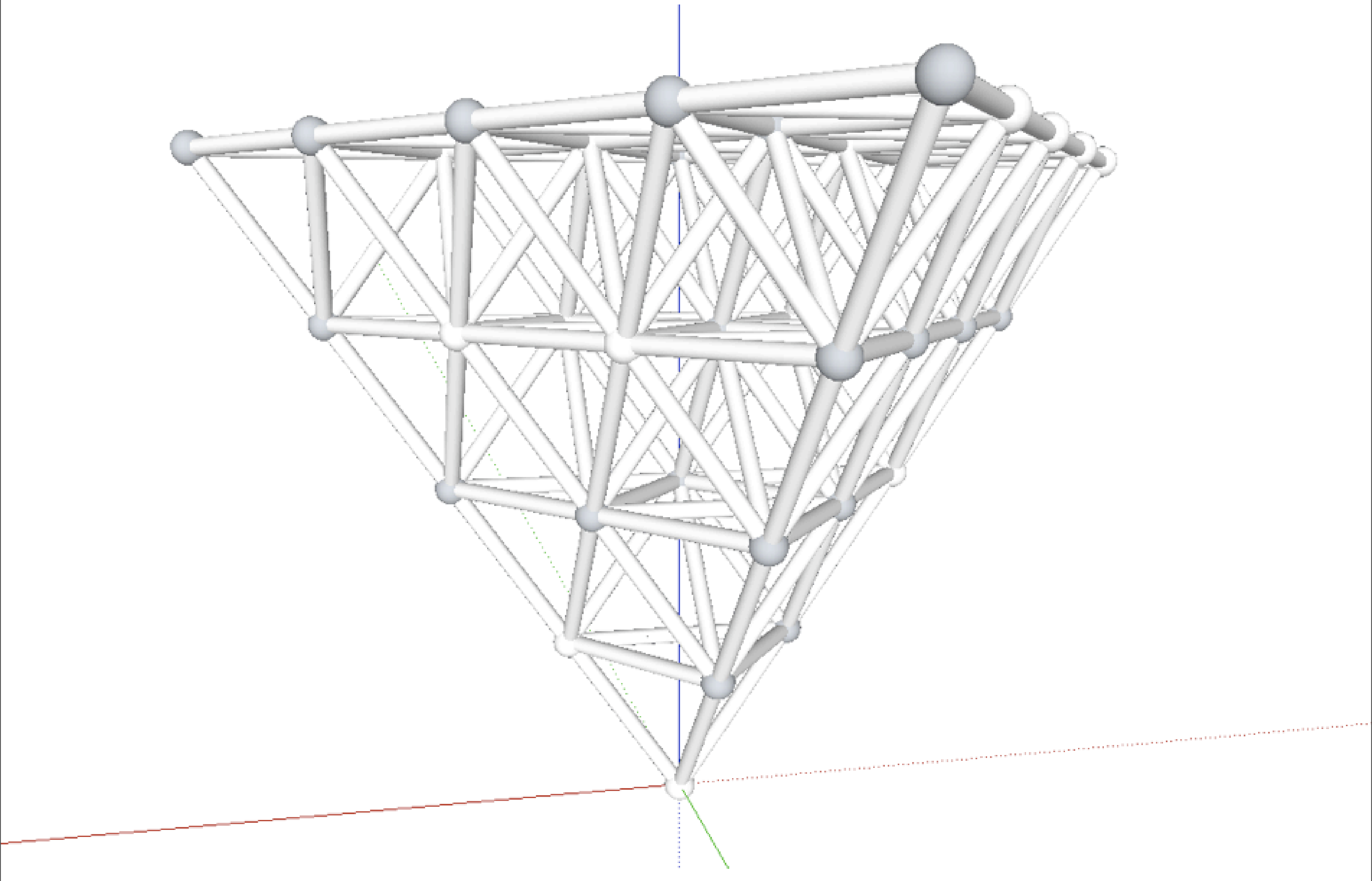
Weak order on 4-cores

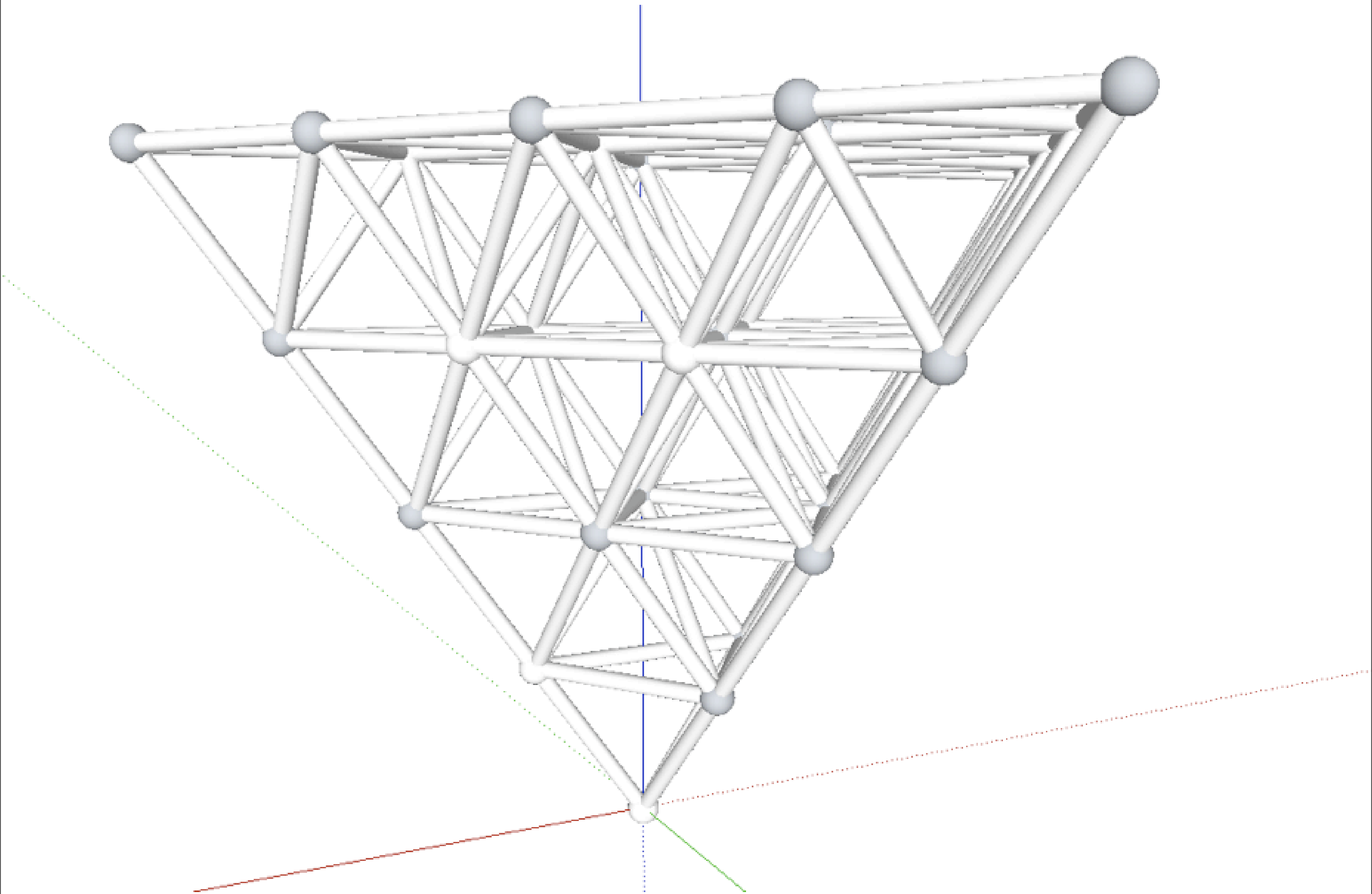


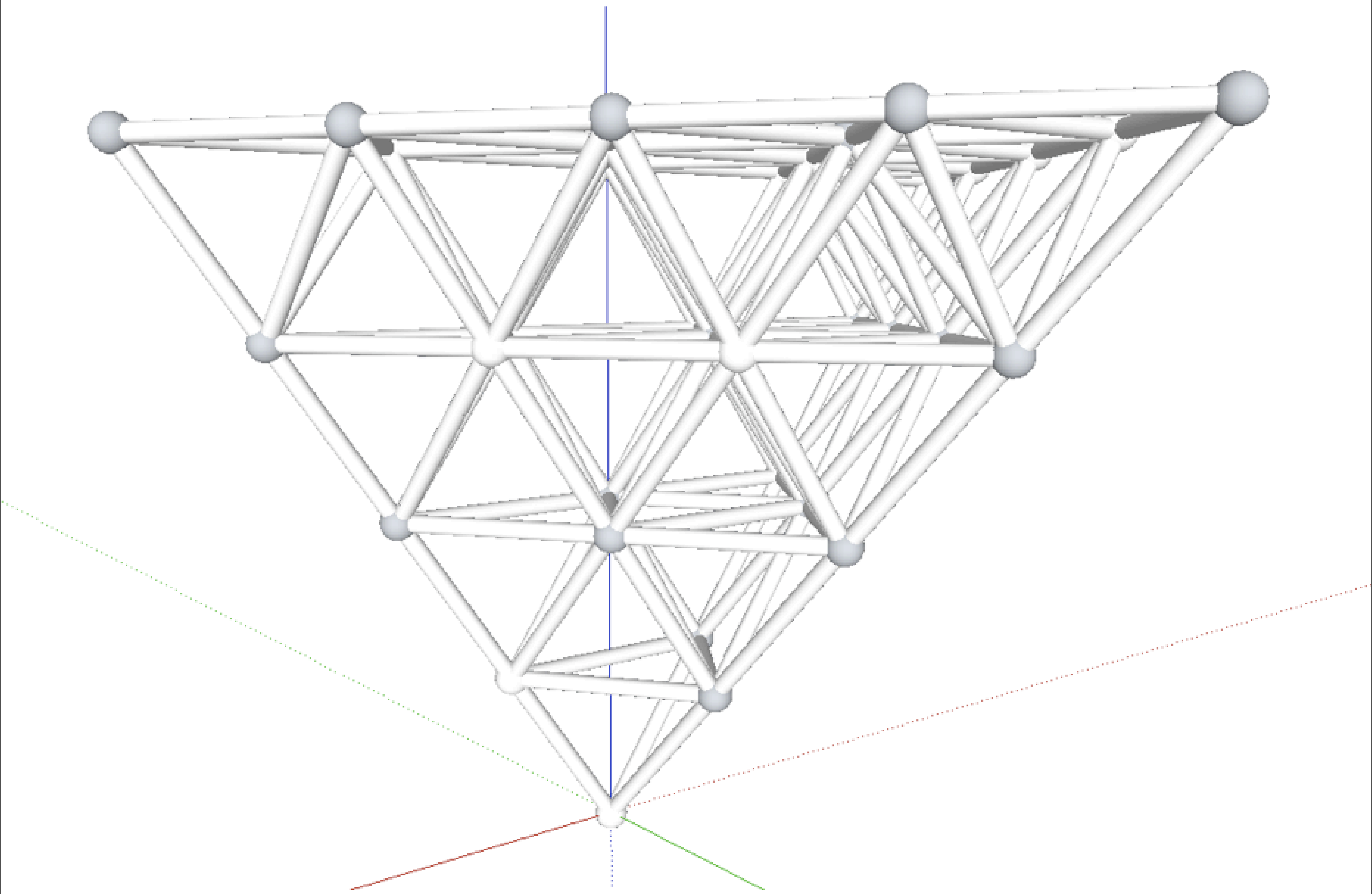


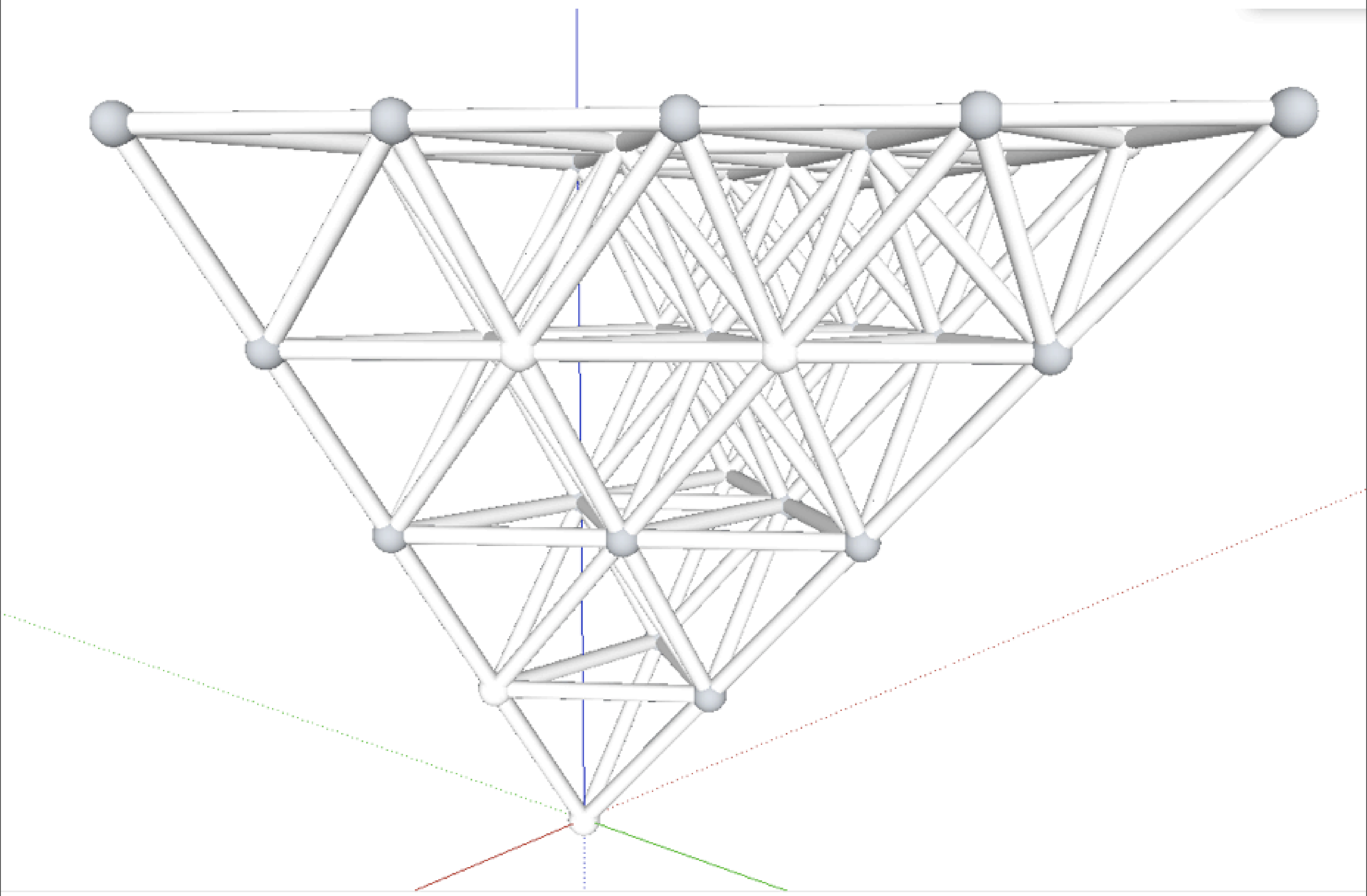


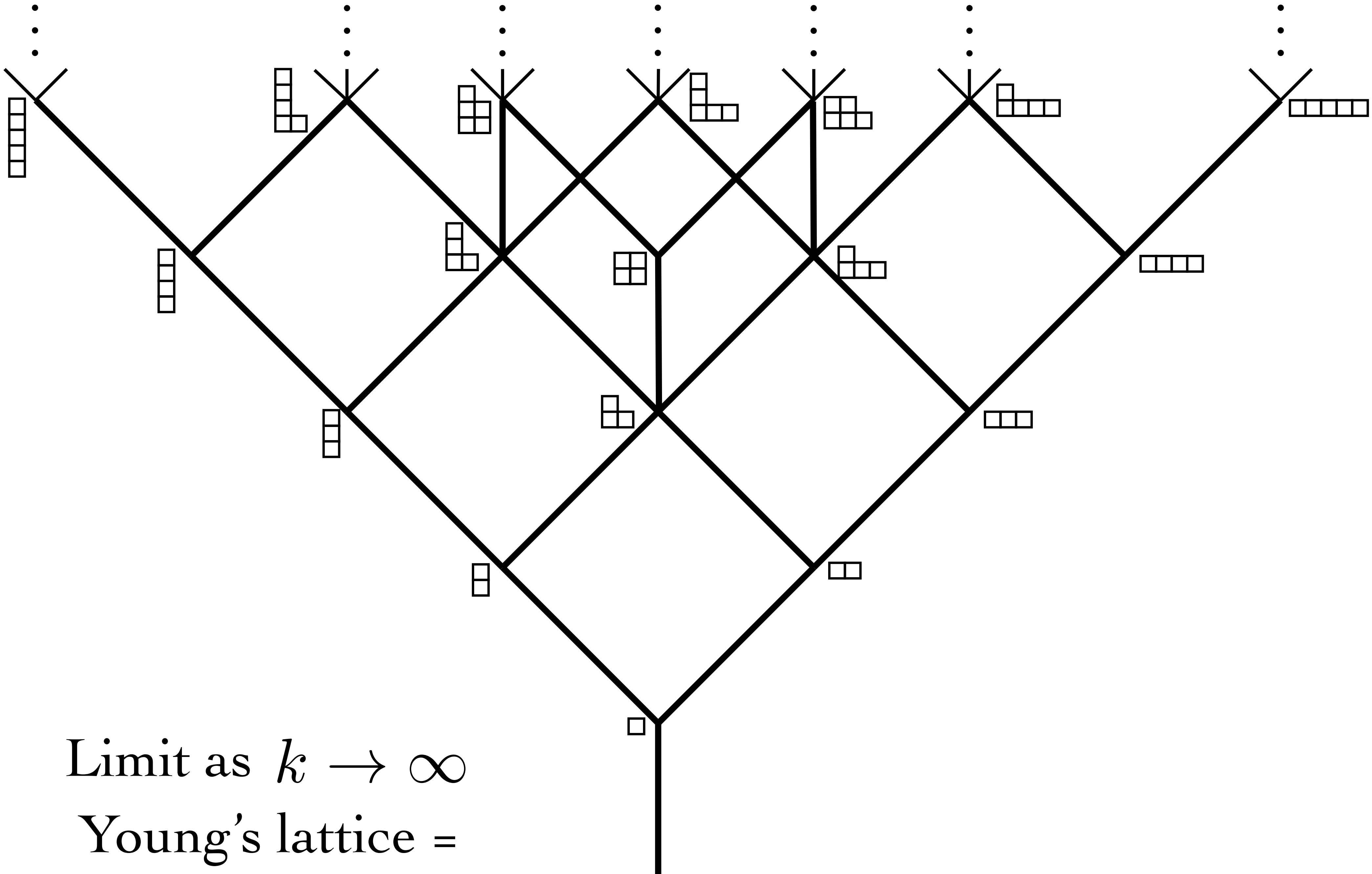












Limit as $k \to \infty$

Young's lattice =

order by inclusion

Thomas Lam

Consider elements of the affine nil-Coxter algebra

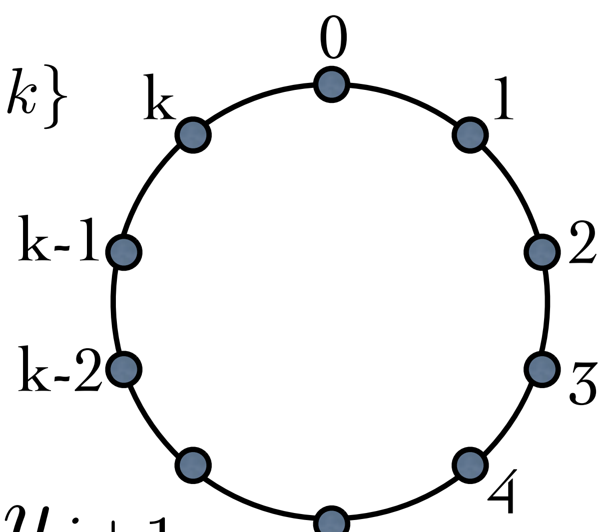
$$u_i^2 = 0$$

$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \qquad i, i+1 \ (mod\ k+1)$$

$$u_i u_j = u_j u_i \qquad i - j \not\equiv \ k, 0, 1 \ (mod\ k+1)$$

$$\mathbf{h}_r = \sum_{|A|=r} u_A \qquad 1 \leq r \leq k$$

$$A \subseteq \{0, 1, 2, \ldots k\}$$

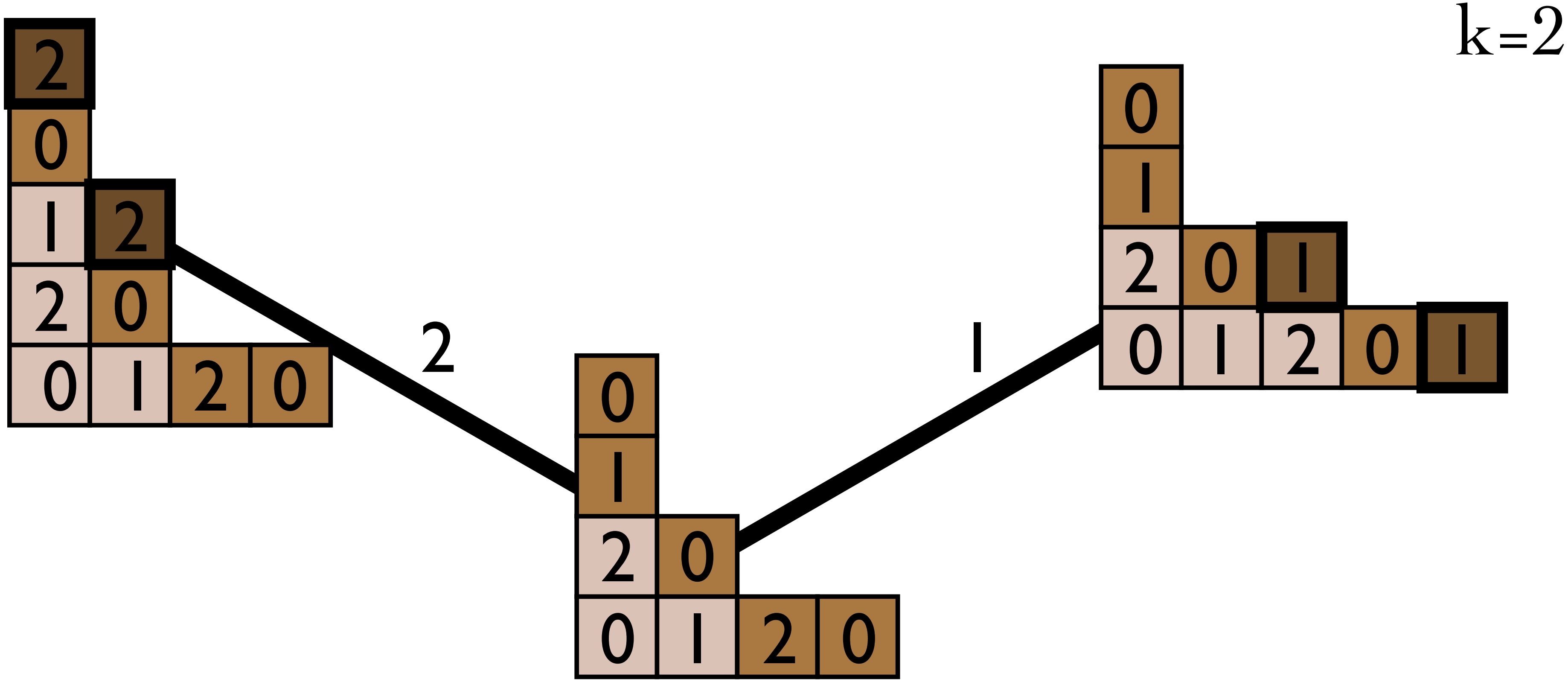


u_A cyclically decreasing word with content A

if $i, i+1 \in A$, u_i comes before u_{i+1}

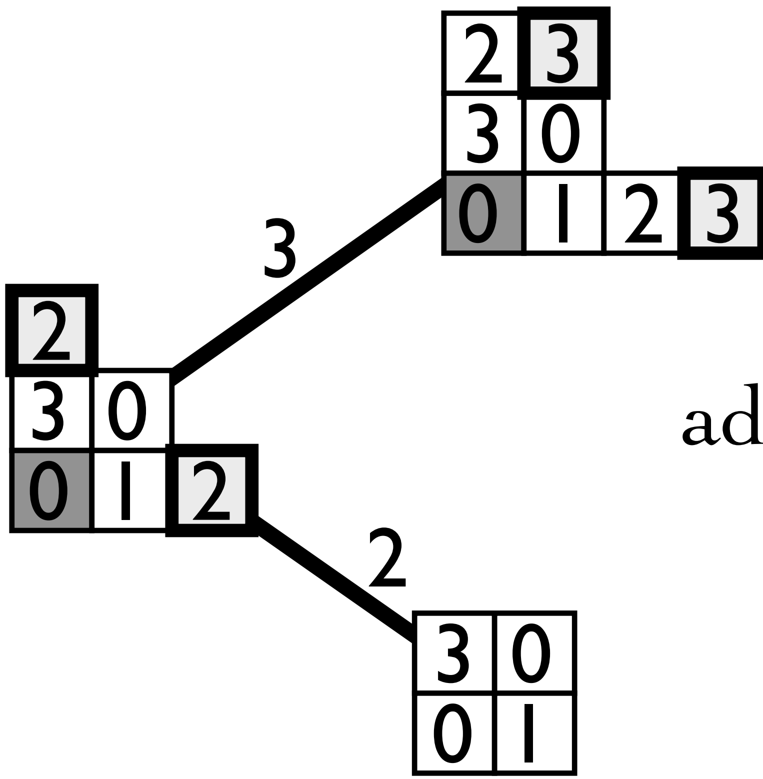
Let γ be a (k+1)-core

u_i acts on γ by adding i-addable corner if possible



the result is 0 otherwise

u_A cyclically decreasing word
with content
if $i, i+1 \in A$ u_i comes before u_{i+1}



k=3

adds a horizontal strip

acting by all cyclically decreasing words adds all possible horizontal strips

$$\mathbf{h}_r(\gamma) = \sum_{\nu} \nu$$

$$\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k] \simeq \mathbb{Q}[\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k]$$

$$u : \mathbb{Q}[h_1, h_2, \ldots, h_k] \to \mathbb{Q}[\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k]$$

$$\mathbf{s}_\lambda^{(k)} = u(s_\lambda^{(k)})$$

Say then that we determine:

$$\mathbf{s}_\lambda^{(k)} = \sum_w c_w w$$

w is in the affine Nil-Coxeter algebra

c_w coefficients

k-Littlewood-Richardson coefficients:

$$\mathbf{s}_\lambda^{(k)} \mathbf{s}_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^{\nu(k)} \mathbf{s}_\nu^{(k)}$$

Viewing this in terms of actions on cores:

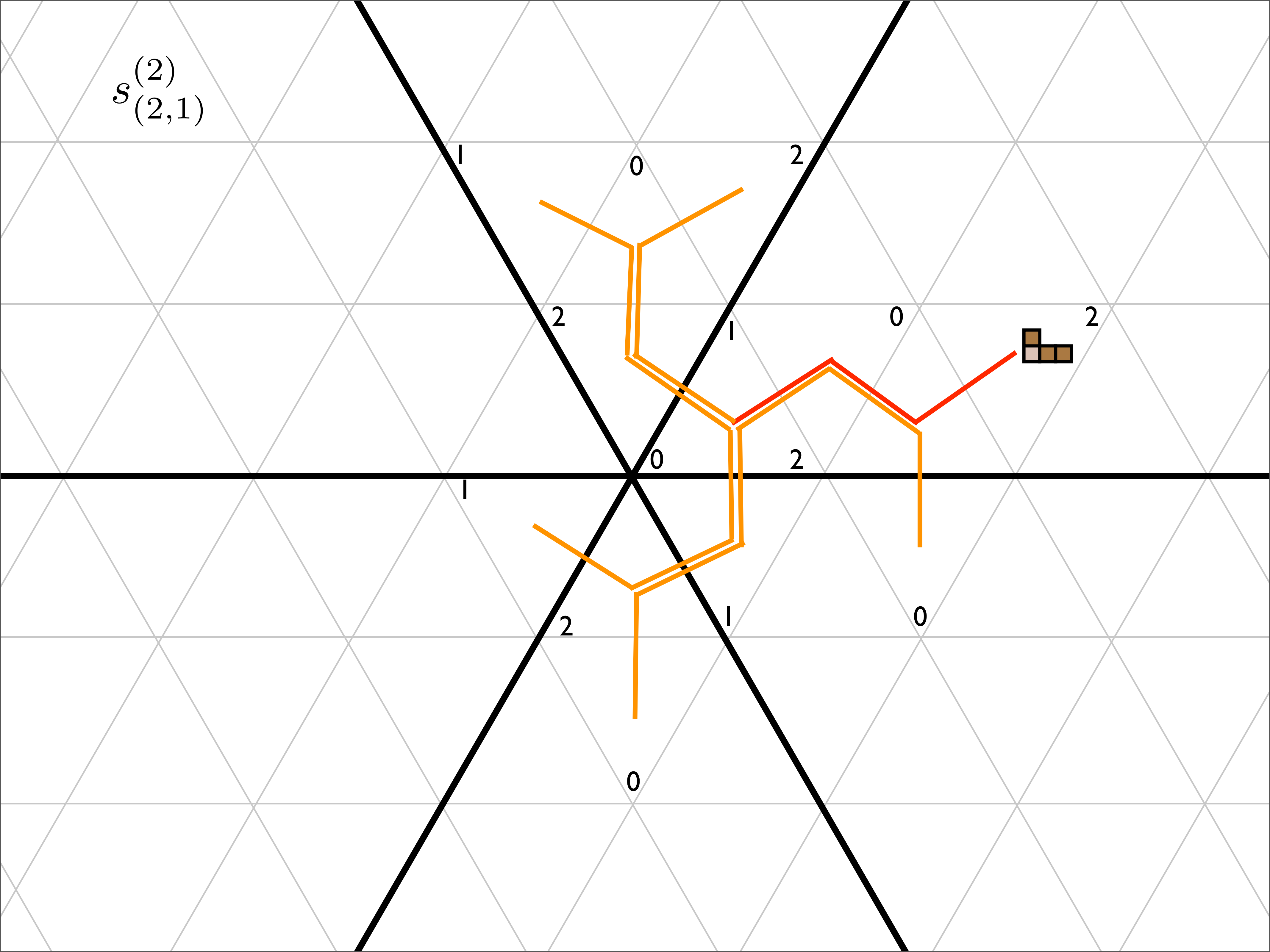
$$\mathbf{s}_\mu^{(k)} \emptyset = \mathfrak{c}(\mu)$$

$$\mathbf{s}_\lambda^{(k)} \mathfrak{c}(\mu) = \sum_\nu c_{\lambda\mu}^{\nu(k)} \mathfrak{c}(\nu) \quad \text{with} \quad \mathbf{s}_\lambda^{(k)} = \sum_w c_w w$$

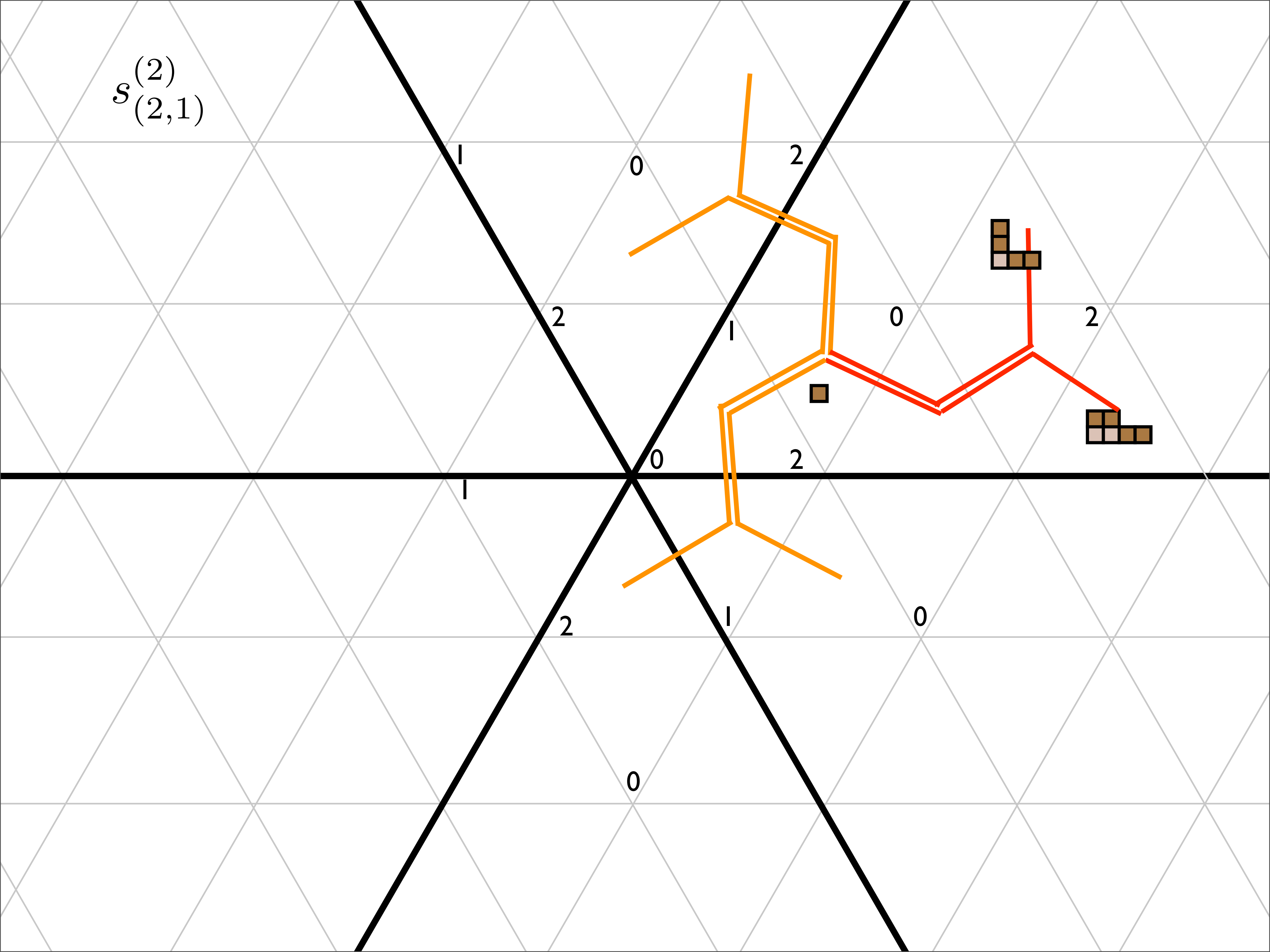
$c_{\lambda\mu}^{\nu(k)}$ is equal to c_w if there exists a w s.t.

$$w\mathfrak{c}(\mu) = \mathfrak{c}(\nu)$$

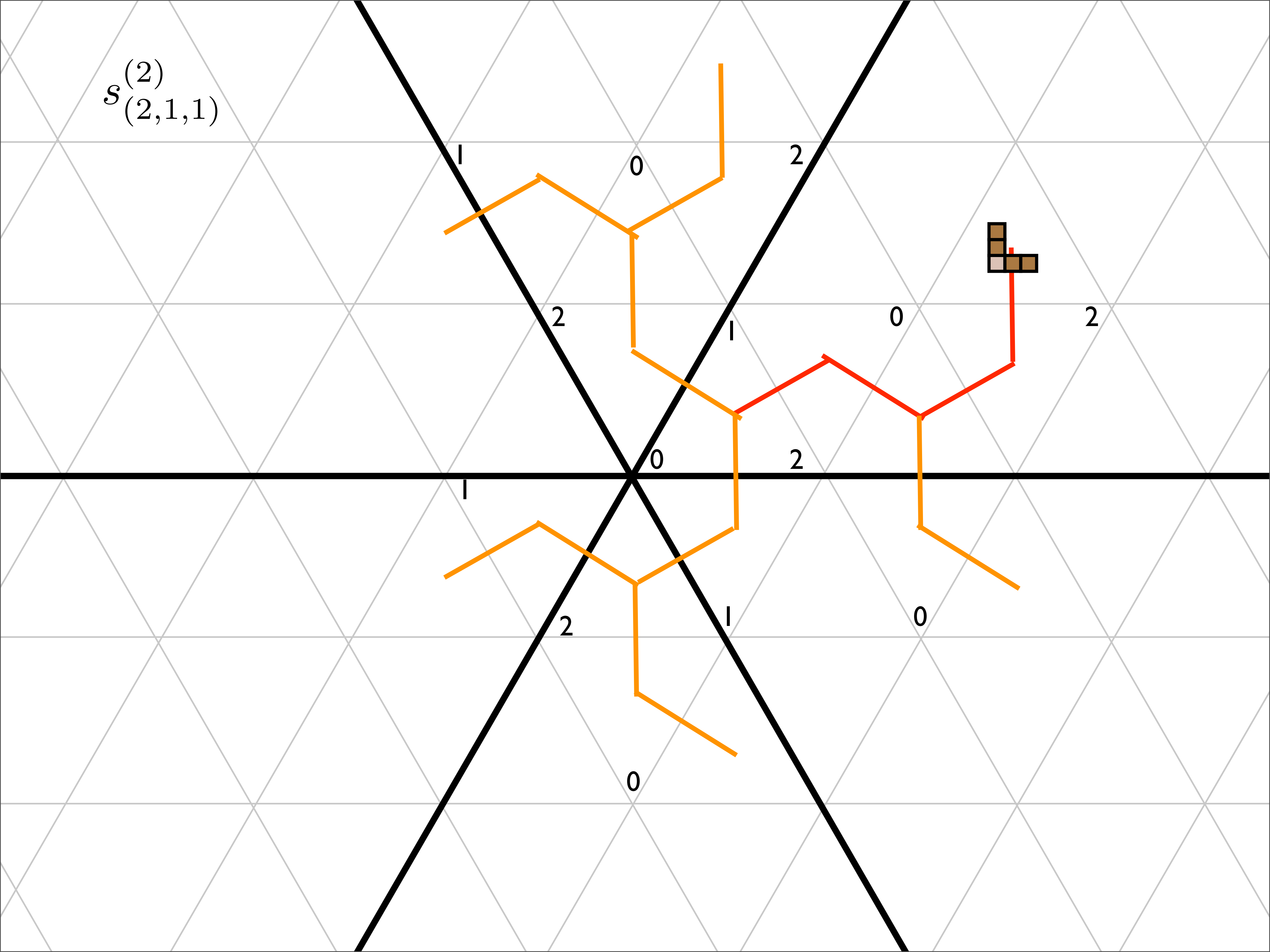
$s^{(2)}_{(2,1)}$

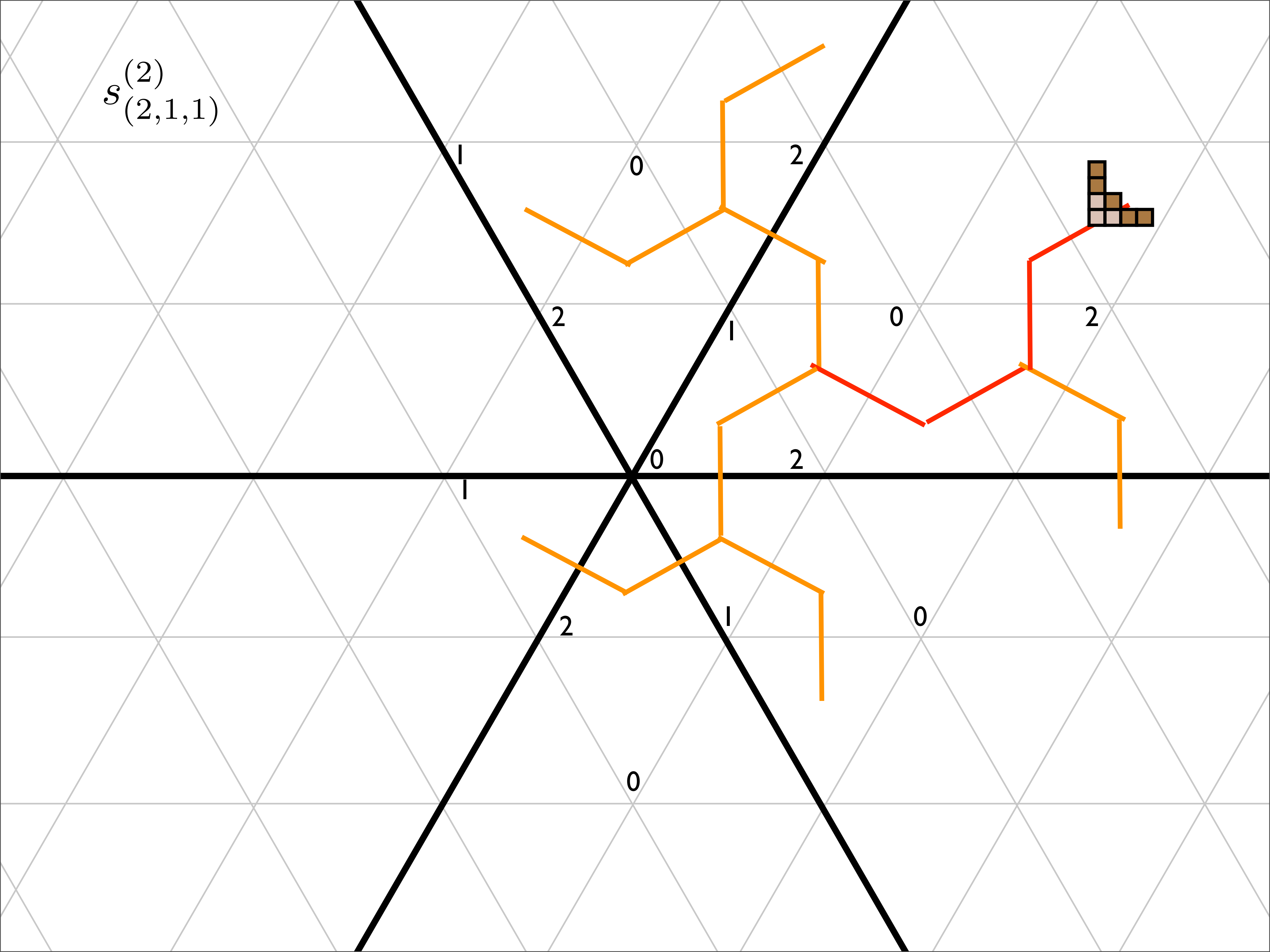


$s^{(2)}_{(2,1)}$



$s^{(2)}_{(2,1,1)}$





$s^{(2)}_{(2,1,1)}$
1
0
2
2
1
0
2
0
2
1
2
1
0
0

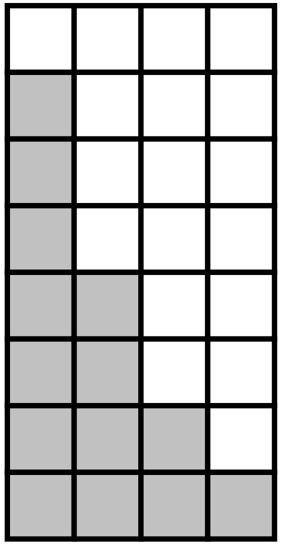
We haven't come up with a k-LR rule, but can reduce it to a more manageable problem

Let R be a rectangle with hook = k

$$s_R s^{(k)}_{\lambda} = s^{(k)}_{R\cup\lambda}$$

$$s^{(k)}_{\lambda} = s_{R_1} s_{R_2} \cdots s_{R_d} s^{(k)}_{\tilde{\lambda}}$$

where each of the R_i are rectangles with hook = k and the partition $\tilde{\lambda}$ contains less than k+2-r parts of size r



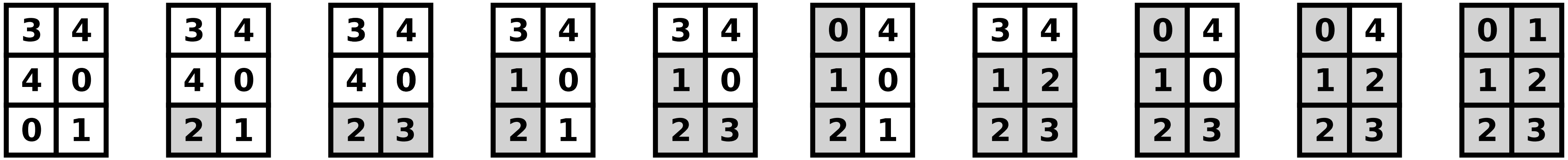
combinatorial formula #1

$$R = ((k+1-r)^r)$$

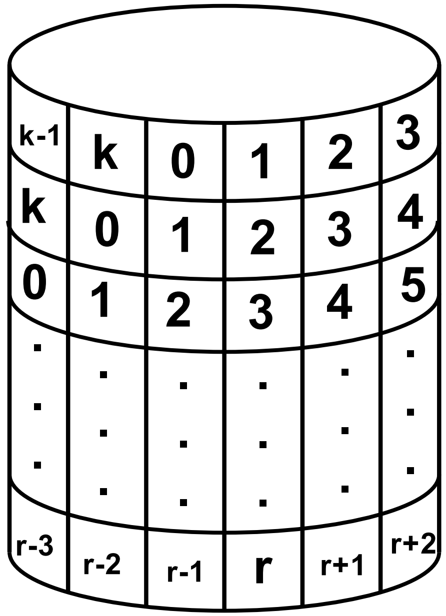
v_λ is a reading of the
grey = contents + k+1-r
white = contents

$$s_R = \sum_{\lambda \subseteq R} v_\lambda$$

Example: k=4 R=(2,2,2)



$$u_4u_3u_0u_4u_1u_0 + u_2u_4u_3u_0u_4u_1 + u_3u_2u_4u_3u_0u_4 + u_1u_2u_4u_3u_0u_1 + u_1u_3u_2u_4u_3u_0$$
$$+u_2u_1u_3u_2u_4u_3 + u_0u_1u_2u_4u_0u_1 + u_0u_1u_3u_2u_4u_0 + u_0u_2u_1u_3u_2u_4 + u_1u_0u_3u_1u_3u_2$$



combinatorial formula #2

$$R = ((k+1-r)^r)$$

$$\mathbf{s}_R = \sum_{|A|=k+1-r} u_A u_{A+1} u_{A+2} \cdots u_{A+r-1}$$

Example: k=4 R=(2,2,2)

0	**1**	2	3	4
1	**2**	3	4	0
2	**3**	4	0	1

0	1	**2**	3	4
1	2	**3**	4	0
2	3	**4**	0	1

0	1	2	**3**	4
1	2	3	**4**	0
2	3	4	**0**	1

0	1	2	3	**4**
1	2	3	4	**0**
2	3	4	0	**1**

0	**1**	**2**	3	4
1	**2**	**3**	4	0
2	**3**	**4**	0	1

0	**1**	2	**3**	4
1	**2**	3	**4**	0
2	**3**	4	**0**	1

0	**1**	2	3	**4**
1	**2**	3	4	**0**
2	**3**	4	0	**1**

0	1	**2**	**3**	4
1	2	**3**	**4**	0
2	3	**4**	**0**	1

0	1	**2**	3	**4**
1	2	**3**	4	**0**
2	3	**4**	0	**1**

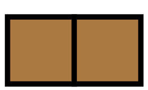
0	1	2	**3**	**4**
1	2	3	**4**	**0**
2	3	4	**0**	**1**

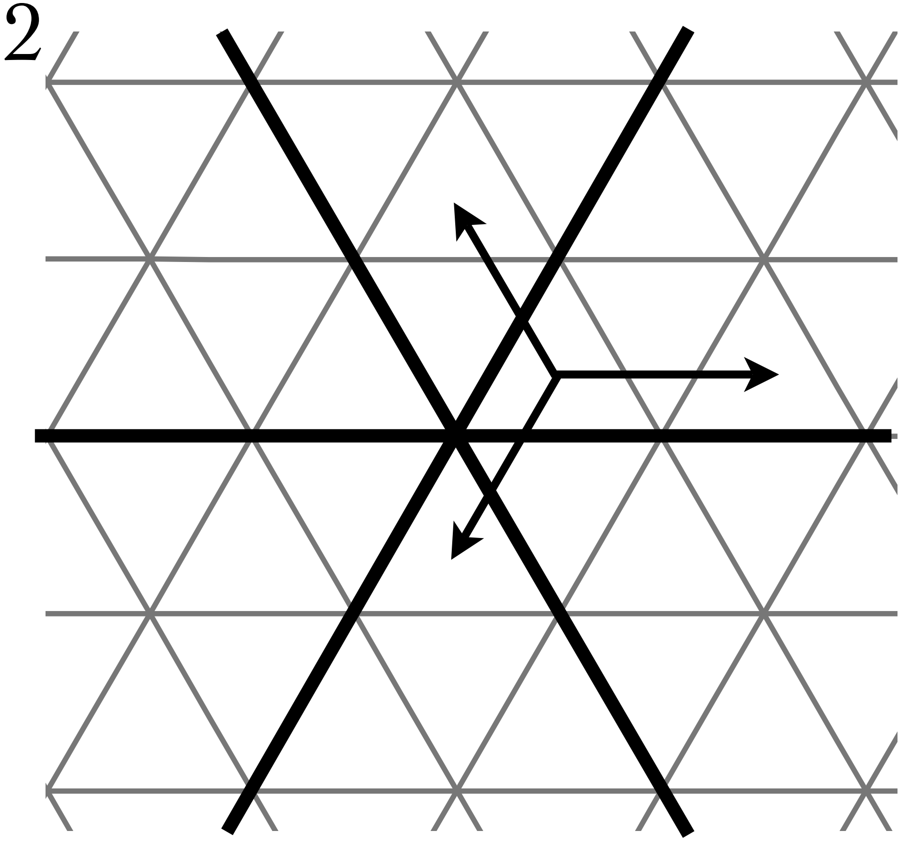
More geometric formula

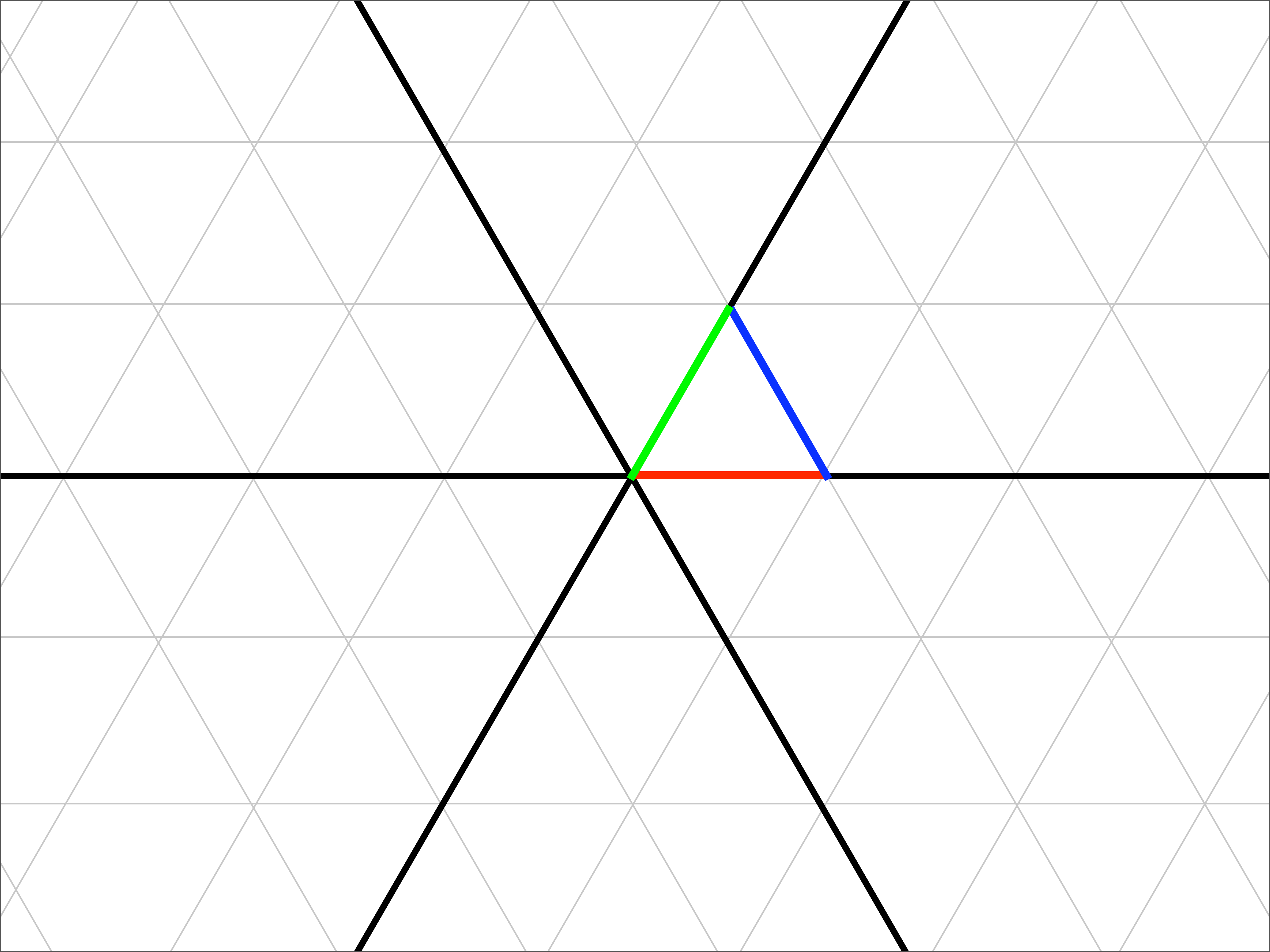
$$\Gamma = \{(a_1, a_2, \ldots, a_{k+1}) : a_i \in \{0,1\}, \sum a_i = k+1-r\}$$

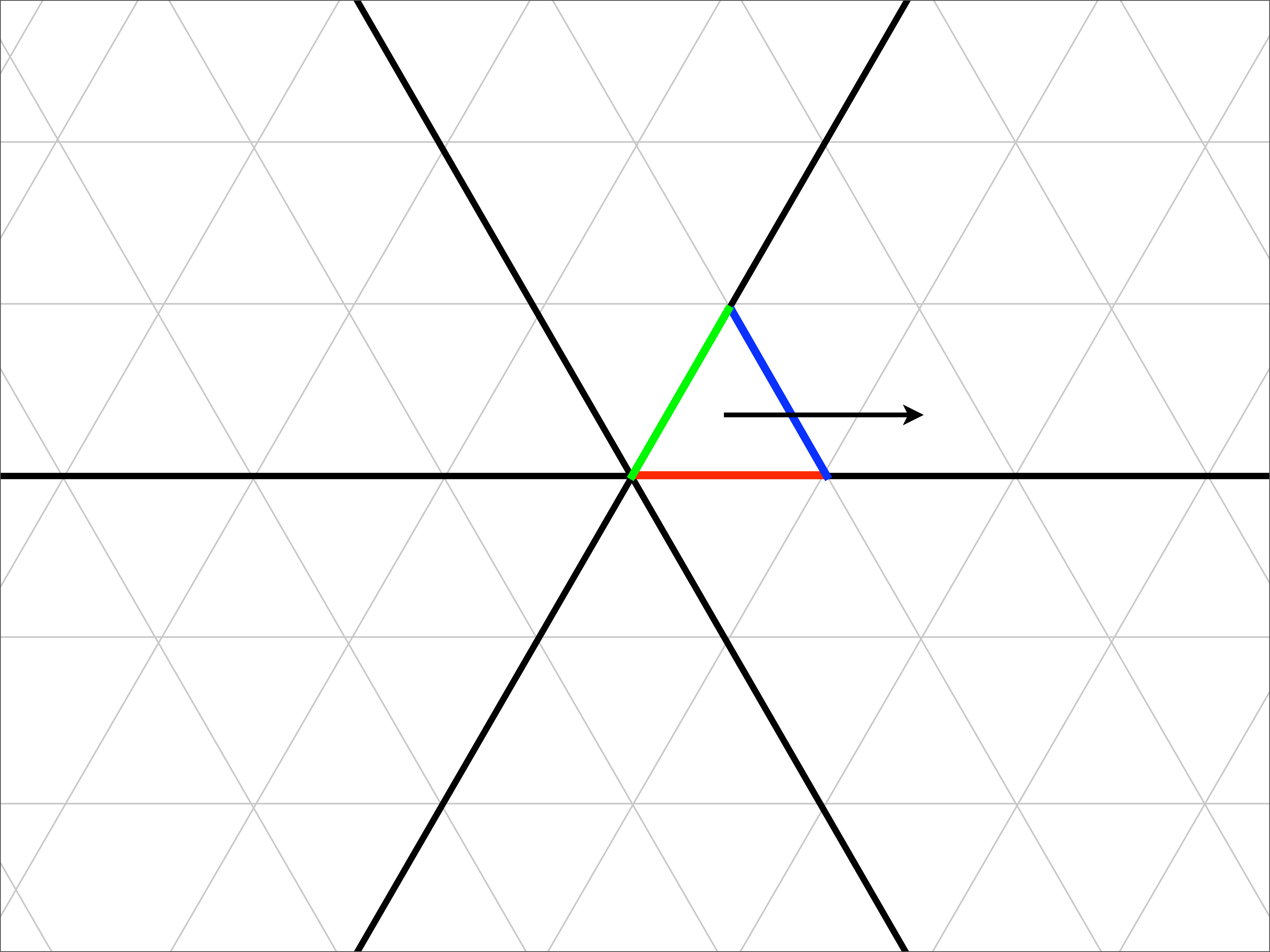
$$s_R = \sum_{\gamma \in \Gamma} \text{pseudo-translation by } \gamma$$

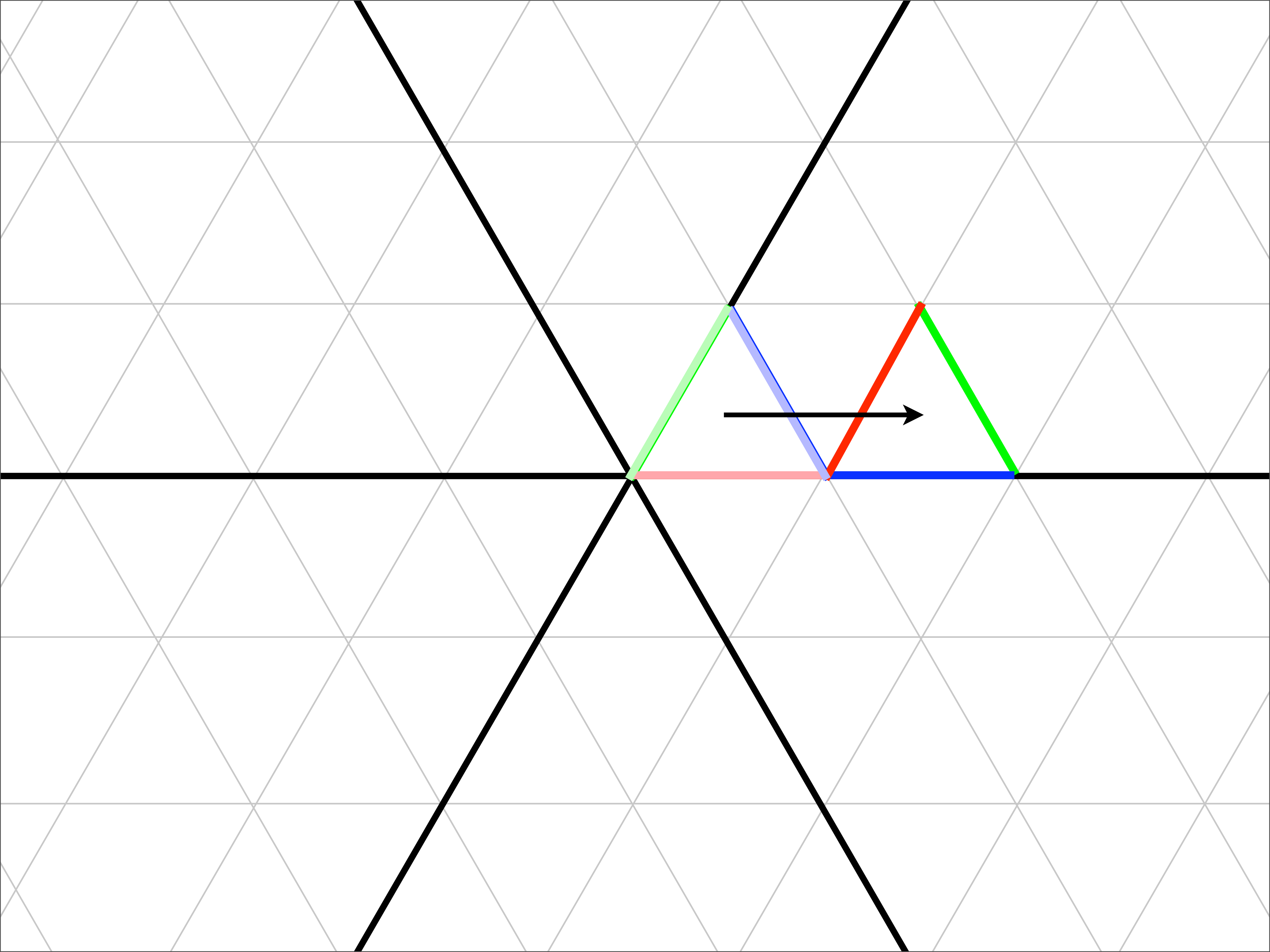
Example: k=2

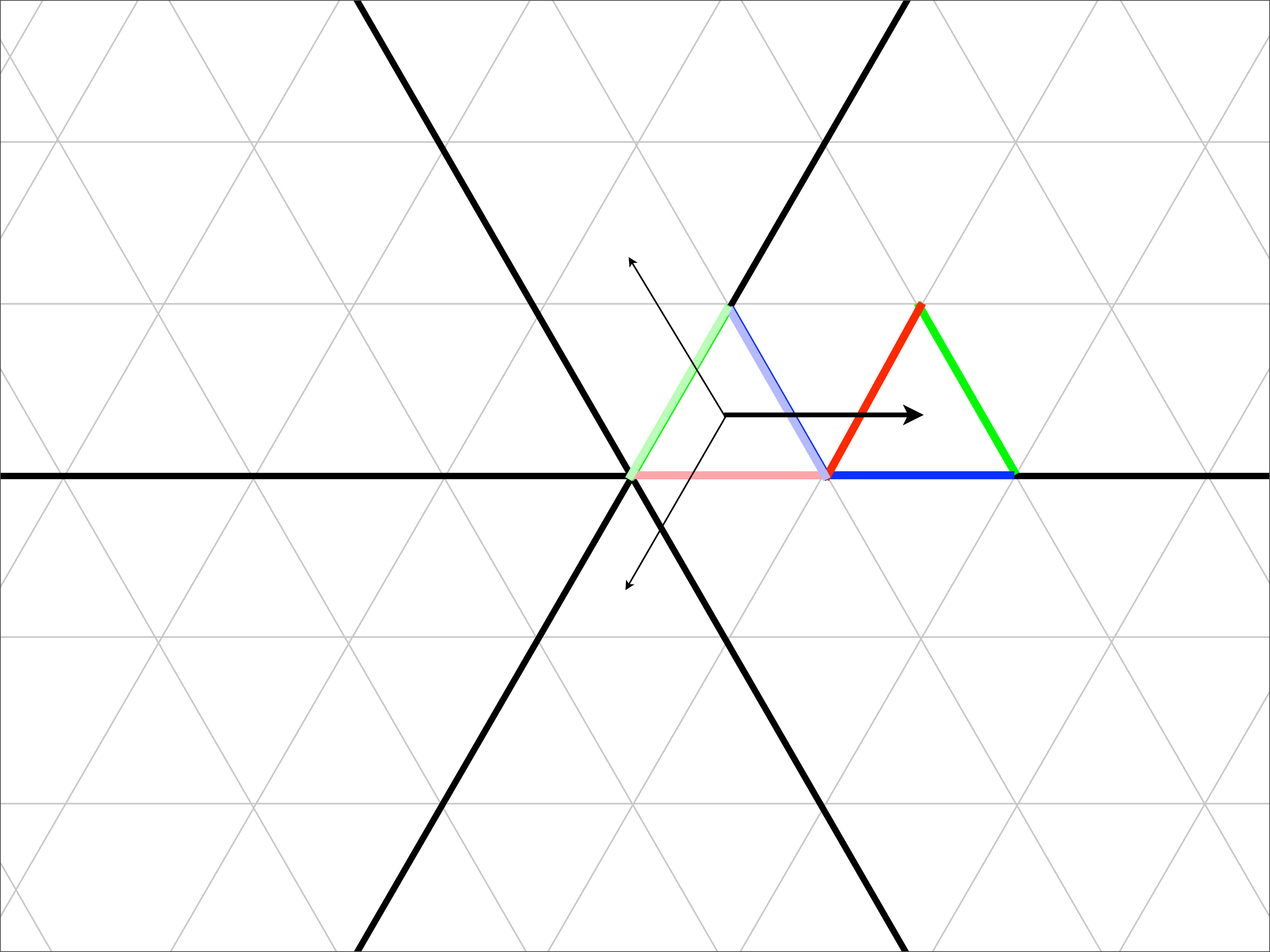
R = 

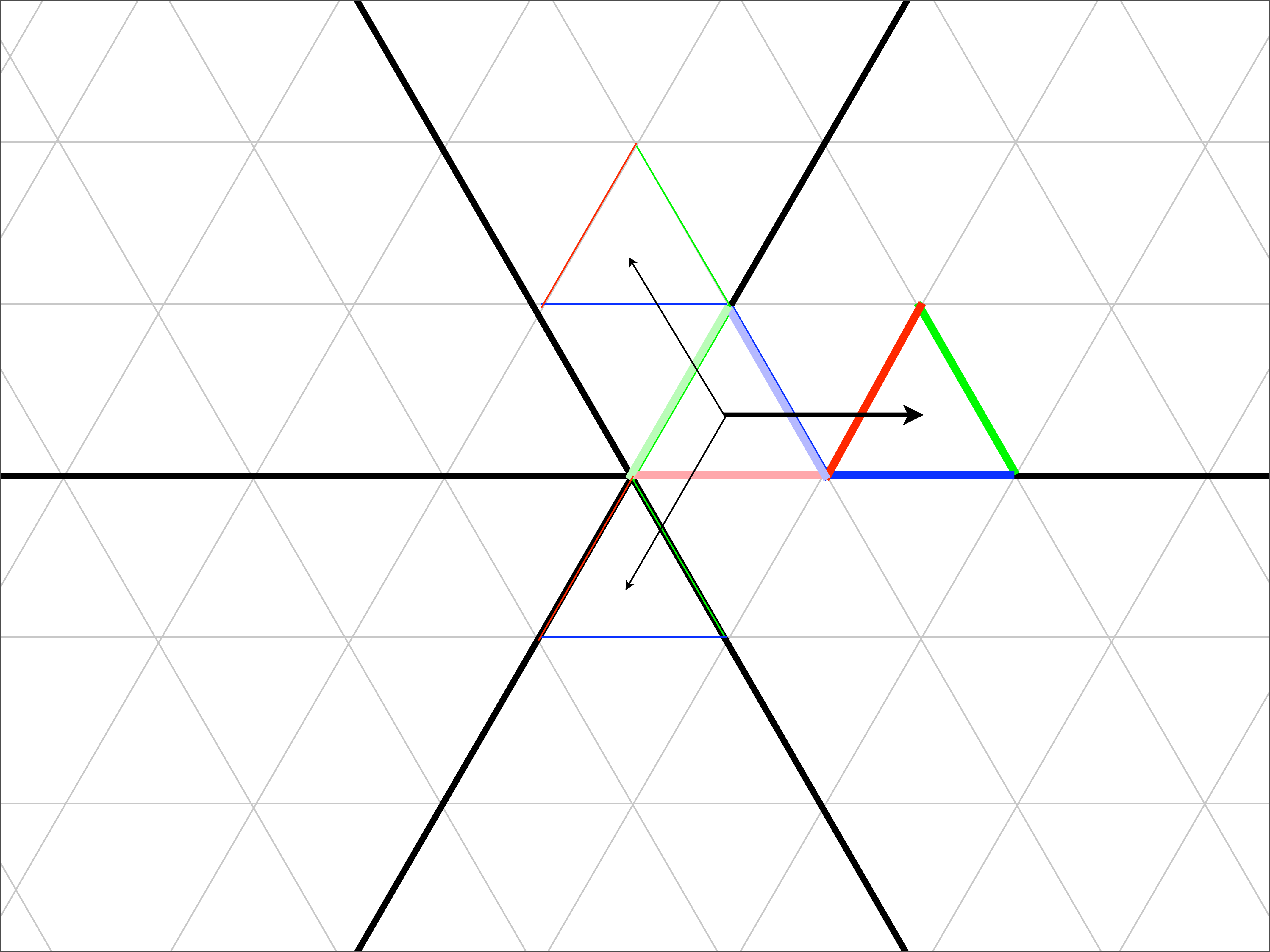


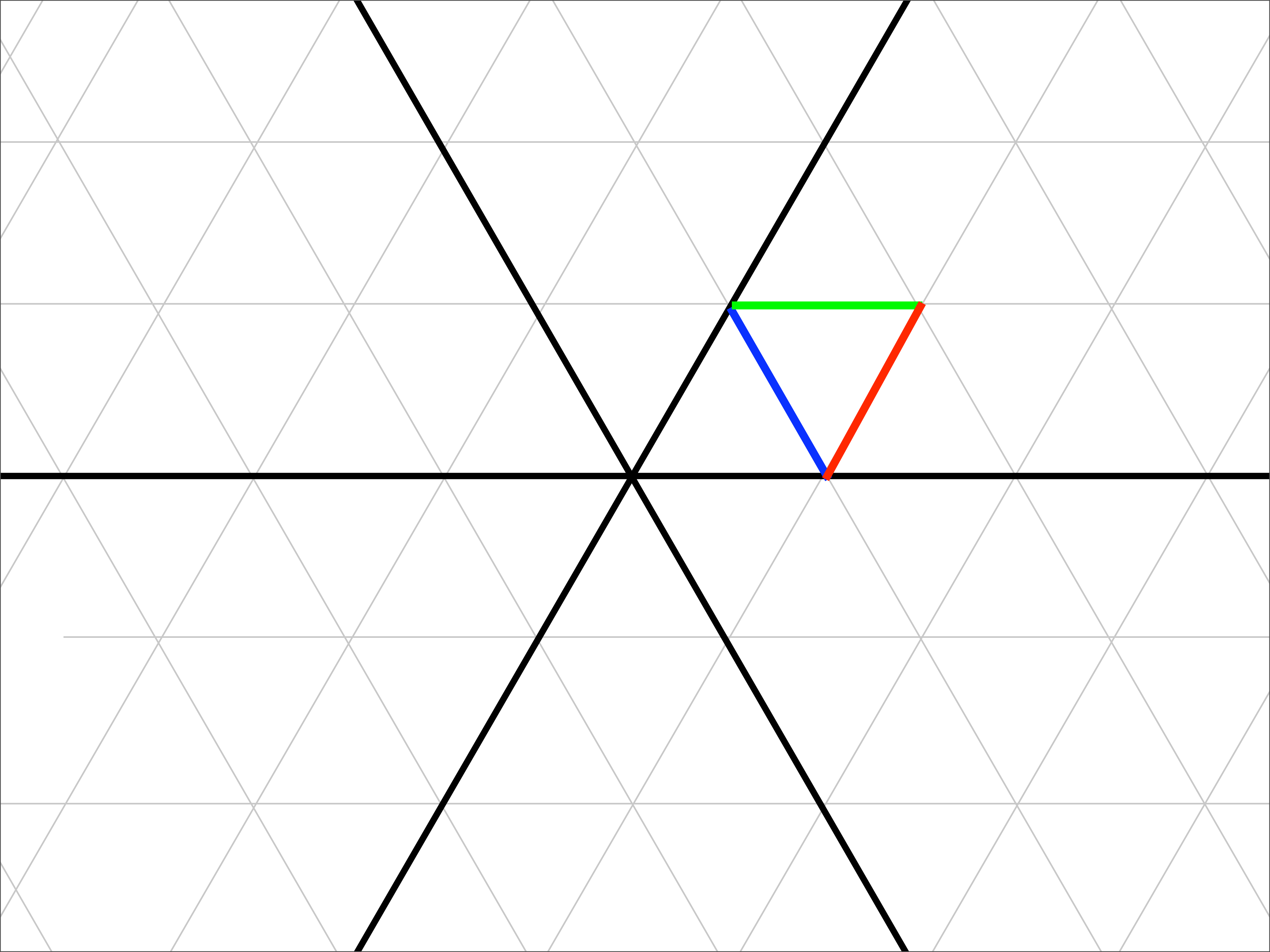


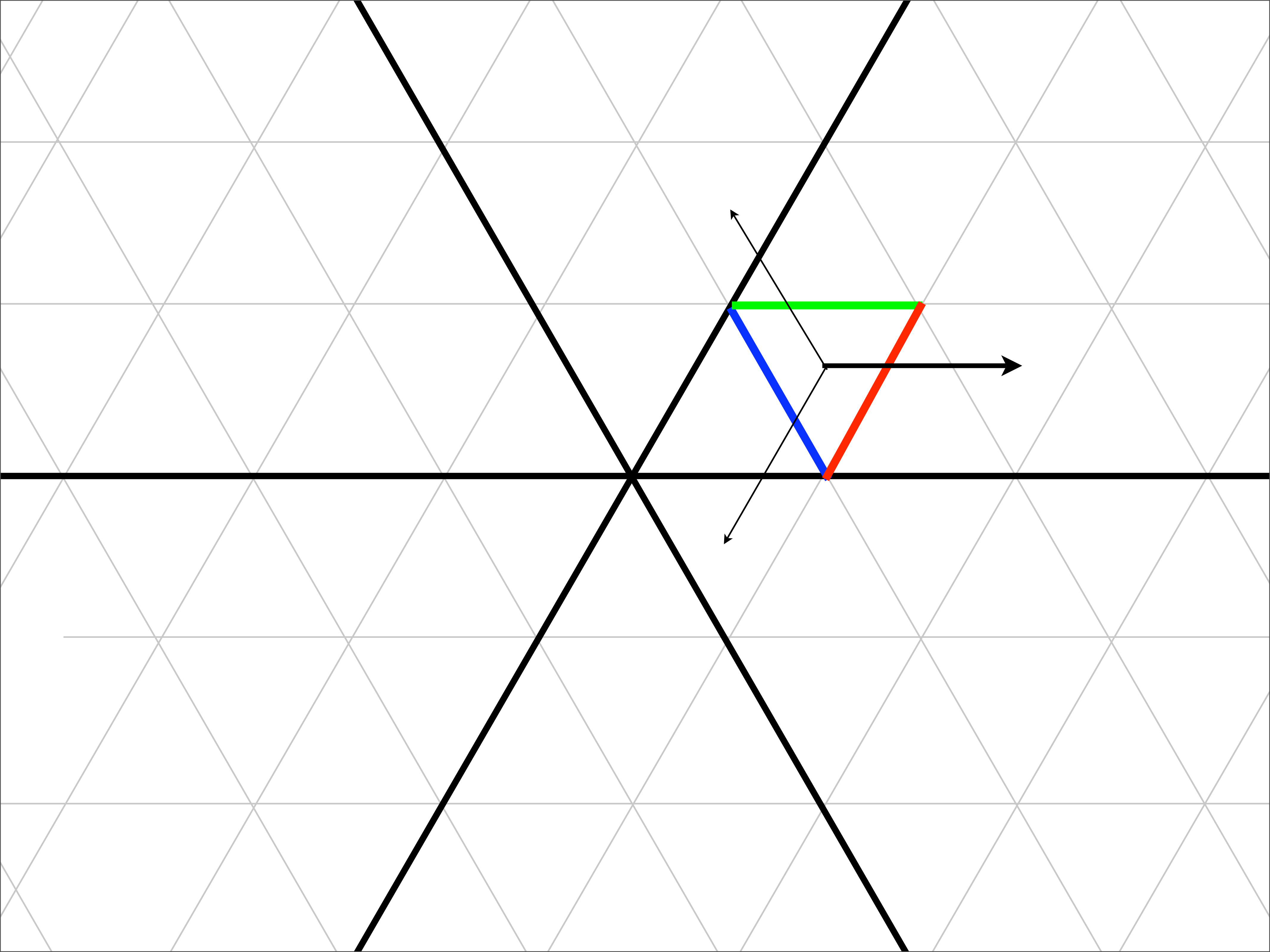


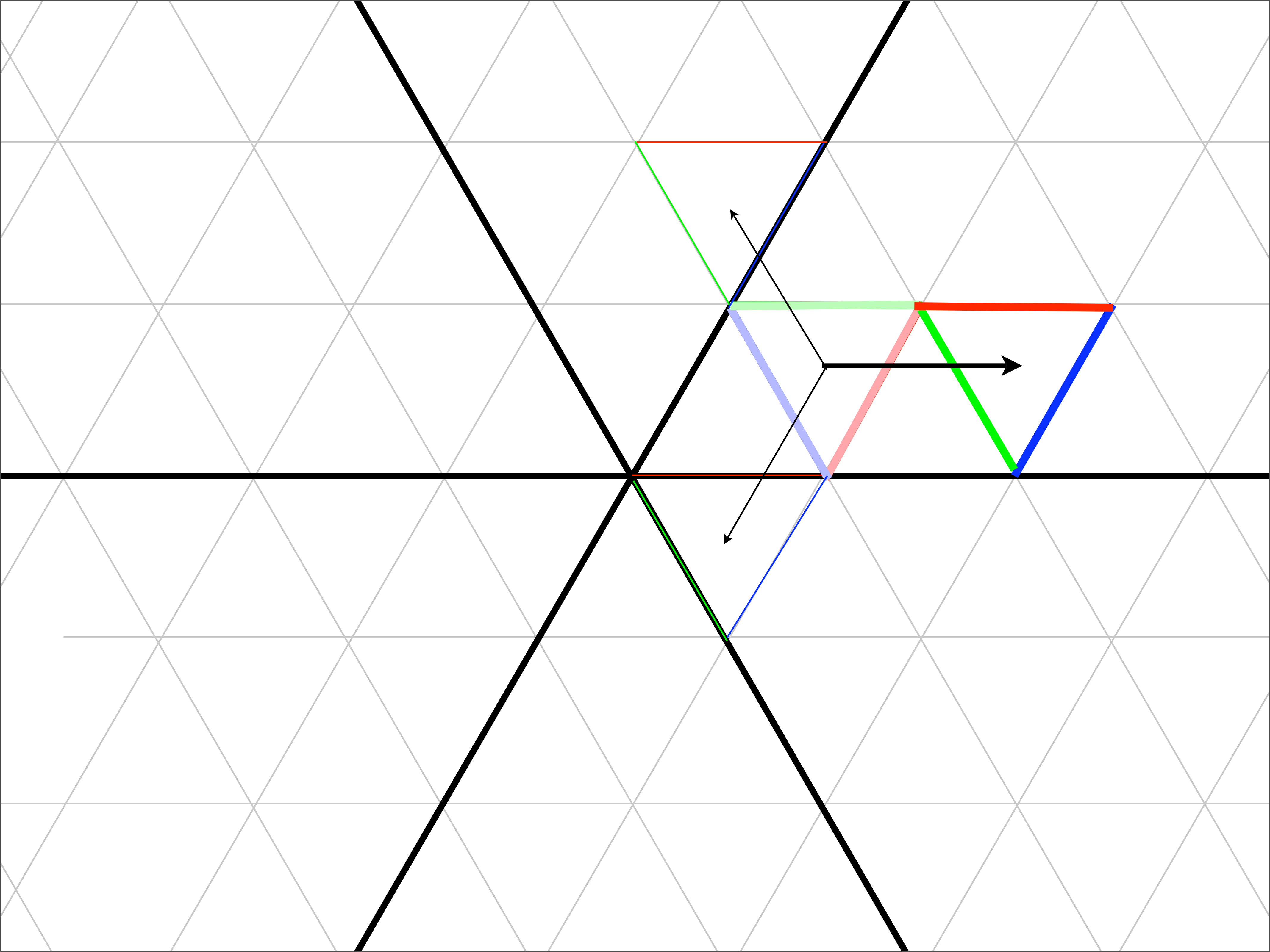












So then what remains to give an explicit k-Littlewood Richardson rule is to give more explicit formulas for $\mathbf{s}^{(k)}_{\tilde{\lambda}}$ where $\tilde{\lambda}$ contains no rectangles with a k-hook.

For a fixed k there are k! such partitions.

END!